*Journal of Advances and Scholarly Researches in Allied Education*
*Vol. 13, Issue No. 2, July-2017, ISSN 2230-7540*

# Properties of Graphical-Algebraic

**Swati Maan***

M.Sc. (Mathematics), CSIR –JRF Qualified

***Abstract – Graphical algebras are both a natural and powerful way to depict intricate dependency structures in multivariate random variables. They come in two flavours, directed or undirected, that are not mutually exclusive. However, some conditional independence structures can only be encoded in one or the other formalism. Among graphical algebras, Gaussian Graphical algebras (often referred to as GGM) are of particular interest because of their ease to be manipulated and interpreted.***

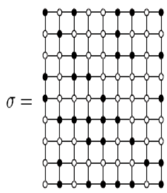
---------------------------------X---------------------------------

## INTRODUCTION

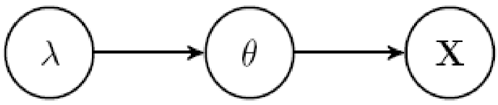
A graphical algebra is a probabilistic model whose conditional (in)dependence structure between random variables is given by a graph. This framework has received a fair amount of attention recently, but the ideas can be traced back as far as the beginning of the XXth century with Gibbs. Indeed, one of the scientific areas that popularised graphical algebras is statistical physics. As an example, let us consider a simple model, named after the physicist Ernst Ising, that can be used to describe a large system of magnetic dipoles (or spins). Spins can be in one of the two states fø1g. They are spread on a graph (commonly a lattice) and can only interact with their neighbours. If ø = (ø1; ø2; ::: ) is a state of the system giving assignments for all spins, the energy of ø and the associated Gibbsian distribution are respectively given by:

$$\mathcal{H}(\sigma) := -J\sum_{i\sim j}\sigma_i\sigma_j - H\sum_i \sigma_i;$$

$$p(\sigma) := \frac{1}{Z}\exp\left(-\mathcal{H}(\sigma)\right).$$




The graphical aspect of this model is rather obvious, since the definition of H depends on the neighbourhood of each spin. This is an example of undirected graphical algebra. Such models are also called Markov random fields. Graphical algebras also naturally arise for instance when designing hierarchical models with sequentially drawn variables. Let us consider a classical Bayesian framework where observations X are drawn according to a distribution with parameters ø, and where ø is itself drawn from a distribution with (hyper)parameters v. This model can be depicted by a directed graph.




Here we have an example of directed graphical model. The idea is that the graph indicates a way to factorise the joint probability distribution of all variables as a product of conditional probability distributions. For this factorisation to be possible, the graph cannot have any directed cycle. It has to be a directed acyclic graph (DAG). These models are often referred to as Bayesian networks or belief networks. Other classical examples include Markov chains and hidden Markov models (HMM).

## GRAPHICAL-ALGEBRAIC PROPERTIES

There are plenty of geometric conditions on a directed graph E which guarantee that L(E) has a particular algebraic structure. For example, we will show that if E is finite and acyclic, then L(E) is a direct sum of matrix rings, and the dimension of each summand can be predicted. Other properties we can characterize \graphically" include: simplicity, primitivity, primality, and pure infiniteness, each of which will be introduced in their respective sections. Most theorems studied in this section will be of the form "E has graphical property (X) ↔ L(E) has algebraic property (Y)."

Many of these properties also hold for C*-algebras. One point of distinction is that prime =) primitive" for separable C*-algebras, but not Lpa's. Another highlight of this section is the dichotomy of simple Lpa's: if L(E) is simple, then it is either locally matricial or purely infinite. An analogous dichotomy holds for graph C*-algebras.

### Ideal structure

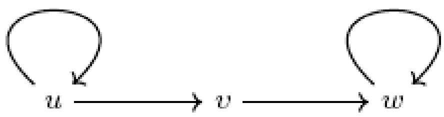
Let L(E) be a Leavitt path algebra. Recall that an ideal I of L(E) is graded if I = Ln2Z I\ L(E)n, where L(E)n denote the homogeneous components of L(E). In this

section we will derive a bijection between the graded ideals of L(E) and certain subsets of E0, called hereditary saturated sets. In case E satisfies \Condition (K)", it turns out all ideals are graded, thus entailing a necessary-and-suficient graphical condition for simplicity of an Lpa. As a consequence of all this, we will be able to prove that the Jacobson radical of L(E) is zero, i.e. all Lpa's are semiprimitive. For C*-algebras, the aforementioned bijection is between hereditary saturated sets and gauge-invariant ideals of Cfi(E) | ideals which are invariant under the gaugeaction . The remark about Condition (K) still holds for C*-algebras. Since all C*-algebras are semiprimitive, the corresponding property for Lpa's is more notable. All the proofs for C*-algebras are the same as in the Lpa case, mutatis mutandis, the only diferences being the application of the gauge-invariant uniqueness theorem versus the graded uniqueness theorem. Thus we only establish the results for Hereditary saturated sets. Let E be a directed graph.
We define two terminologies: a set H ⊆ E0 is

- hereditary if: whenever e ∈ E1 and s(e) ∈ H, then r(e) ∈ H.
- saturated if: whenever v ∈ E0 is a nonsink such that r(e) ∈ H for all e ∈ s−1(v), we must have v ∈ H.

Heredity can be interpreted as saying that once you are in H, you cannot get out. Saturation is the converse: if you only emit edges leading into H, then you must have been in H to begin with. So a hereditary saturated set can be thought of as being trapped in an exclusive zombie night club | if all your friends are in, the bouncers will let you in too; however, once you're in you will never escape.

Example. Consider the following graph:




The set fv;wg is hereditary and saturated; note that u has an outneighbor in fv;wg, but not all outneighbors are in fv;wg, so the saturation condition does not fail at u. The set fug is saturated but not hereditary, the set fwg is hereditary but not saturated, and the set fu;wg is neither hereditary nor saturated. Punchline: \hereditary" and \saturated" are mutually exclusive conditions. Hereditary sets are important because they generate subgraphs (see 1.2.4): if H is hereditary, we define a subgraph E n H, called the complementary subgraph, by deleting H and all edges leading into it.

E \ H := E0 \ H, r−1(E0 \ H), r, s

Since any edge of E \ H must end outside H, it must also start outside H by heredity. So E\H is indeed a subgraph in the sense of 1.2.4.

Example. In the graph of the previous example, we have the following subgraphs obtained by removing the sets {w} and {v, w}.


3.1.2 Ideals → hereditary saturated sets. Now we will begin to correspond ideals to hereditary saturated sets of vertices. For an ideal I of L(E), consider the set I ∩ E0 = {v ∈ E0 : v ∈ I}.

Proposition. Let I be an ideal. Then I ∩ E0 is hereditary and saturated.

Proof. For heredity, suppose e is an edge with s(e) ∈ I. r(e) = e∗e = e∗s(e)e ∈ I so I ∩ E0 is hereditary. For saturation, suppose v is a nonsink such that r(e) ∈ I whenever s(e) = v. We must check v ∈ I. Indeed, by (CK2) we have

$$v = \sum_{s(e)=v} ee^* = \sum_{s(e)=v} er(e)e^* \in I$$

This is row finite since E is, so we can form the Leavitt path algebra L(EI) over the same field. As we will now show, there is a nice CK EI-family in the quotient Algebra L(E)/I.

Corollary. The map L(EI ) ! L(E)=I given by e 7! e + I and v 7! v + I induces a surjective algebra homomorphism. Consequently, L(EI ) ' L(E)=I if one of the following conditions holds:

(i) EI satisfies Condition (L); or

(ii) I is graded.

We'll see later that (i) implies (ii).

Proof. Let x 7! x denote passage to the quotient L(E)=I; first we check that fege2EI , fvgv2EI satisfy the CK relations. (CK1) is automatic: efie = r(e) since the preimages satisfy (CK1) in L(E). For (CK2), note that if v is a nonsink in EI then it is a nonsink in E, so (CK2) in E implies

$$\begin{aligned}
\bar{v} &= \sum_{\substack{e\in E^1\\ s(e)=v}} \bar{e}\bar{e}^* && \text{by (CK2) in } E\\
&= \sum_{\substack{r(e)\notin H\\ s(e)=v}} \bar{e}\bar{e}^* + \sum_{\substack{r(e)\in H\\ s(e)=v}} \bar{e}\bar{e}^*\\
&= \sum_{\substack{r(e)\notin H\\ s(e)=v}} \bar{e}\bar{e}^* && \text{since } r(e)\in H \implies ee^* = er(e)e^* \in I\\
&= \sum_{\substack{e\in (E_I)^1\\ s(e)=v}} \bar{e}\bar{e}^*
\end{aligned}$$

which is (CK2) in EI . Therefore fege2EI , fvgv2EI satisfy the CK relations, and so by the universal property of L(EI ) there is an algebra homomorphism ' : L(EI ) ! L(E)=I sending e 7! e and v 7! v. This is surjective since this CK EI-family clearly generates L(E)=I.

Note that if v 2 EI , then v =2 H and so v 6= 0. Thus if EI has Condition (L) then ' is an isomorphism by the Cuntz{Krieger Uniqueness Theorem; if I is graded then ' is a graded homomorphism, hence injective by the Graded Uniqueness Theorem. As required. So I ∩ E0 is saturated.

Hereditary saturated sets $ ideals. Now we complete the correspondence between hereditary saturated sets and ideals. If H fi E0 is a hereditary saturated set, we correspond it to an ideal I of L(E) in the naive way, which turns out to work: simply let I be the ideal generated by H.

$$I := \langle v : v \in H \rangle.$$

Since this is generated by homogeneous elements, in fact I is a graded ideal. It turns out that this is the unique graded ideal with I \ E0 = H

**Simplicity**.

Now it is because R = k[x, x−1] r(ei) = s(ei+1) fails Condition (L) as a Leavitt path algebra. We can give graphical conditions characterizing when an Lpa is simple but we have to introduce some graph-theoretic terminology. First, an infinite path in a directed graph E is a sequence of edges μ = e1e2e3 · · · such R = k[x, x−1]for all i; we denote by E1 the set of infinite paths in E. We say E is cofinal if, for all R = e k[x, x−1]2 E1, all sinks w 2 E0 and all vertices v 2 E0, there are paths μ = 1e2e3 · · · and R = k[x, x−1]Succinctly, this means any vertex in E can reach any infinite path and any sink; in particular every vertex can reach every cycleR = k[x, x−1]because R = k[x, x−1]is an μ = e1e2e3 · · · nite path clear that E has no nontrivial hereditary saturated sets if and only if R = k[x, x−1]is graded-simple, i.e. has no nontrivial graded ideals. For example R = k[x, x−1]is graded-simple, which can be seen either directly or from its structure as a Leavitt path algebra. But it is not simple, because e.g. 1 − x)R is a proper nonzero ideal; we will see that this is

cofinal not cofinal

If E is finite, the only infinite paths are ones containing cycles | so in this case, cofinality is equivalent to saying that any vertex can reach any cycle and any sink. Clearly, then, a cofinal graph has at most one sink: if it has two sinks, they can't reach other. The following fact reveals the importance of cofinality: essentially E is cofinal if and only if L(E) is graded-simple.

**Prime and primitive ideals**

The section is structured as follows: first we investigate prime and primitive Lpa's, then prime and primitive C*-algebras. For each, we can find graphical conditions to determine which graded ideals are prime and/or primitive. We will see that Cfi(E) is prime if and only if it is primitive, if and only if L(E) is primitive | but it is possible for L(E) to be prime while Cfi(E) is not. This will be our first example of an Lpa property which does not perfectly reect C*-algebras.

**Prime rings.**

Let R be a ring. An ideal P of R is prime if, whenever I; J are ideals with IJ fi P, either I fi P or J fi P. It is equivalent to require that for all x; y 2 R, xRy fi P implies x 2 P or y 2 P (see Proposition 10.2 in [15]). R is a prime ring if f0g is a prime ideal; that is, if x; y 2 R and xry = 0 for all r 2 R, then either x = 0 or y = 0. If R is commutative and unital, this reduces to the condition that R has no zero divisors, i.e. R is an integral domain. For noncommutative rings it is sufficient but not necessary that R has no zero divisors, e.g. Mn(C) is a prime ring with many zero divisors. We now investigate prime Leavitt path algebras. Proposition. Let L(E) be a Leavitt path algebra. Then L(E) is a prime ring if and only if E is downward directed: that is, for all vertices v;w 2 E0, there exists x 2 E0 such that v ! x and w ! x. Recall that \v ! w" just means that there is a path fi from v to w, i.e. s(fi) = v and r(fi) = w.

Let R be a ring. If M is a right R-module, its kernel (or annihilator) is the two-sided ideal kerM := fr 2 R : Mr = 0g. A proper ideal P of R is (right) primitive if P = kerM for some nonzero, simple right R-module M. If f0g is a primitive ideal we say that R is a (right) primitive ring; equivalently, R admits a faithful simple right R-module. Thus an ideal P is primitive if and only if R=P is a primitive ring. We note that in commutative rings, the only primitive rings are _elds, and the primitive ideals are the maximal ideals. Here is a standard fact.

## REFERENCES

A.I. Shtern (2011). "C* algebra", in Hazewinkel, Michiel, Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4

Arveson, W. (2006). An Invitation to C*-Algebra, Springer-Verlag, ISBN 0-387-90176-0. An excellent introduction to the subject, accessible for those with a knowledge of basicfunctional analysis.

Connes, Alain, Non-commutative geometry (PDF), ISBN 0-12-185860-X. This book is widely regarded as a source of new research material, providing much supporting intuition, but it is difficult.

Dixmier, Jacques (2009). Les C*-algèbres et leurs représentations, Gauthier-Villars, ISBN 0-7204-0762-1. This is a somewhat dated reference, but is still considered as a high-quality technical exposition. It is available in English from North Holland press.

Doran, Robert S.; Belfi, Victor A. (2006). Characterizations of C*-algebras: The Gelfand-Naimark Theorems, CRC Press, ISBN 978-0-8247-7569-8.

Emch, G. (2002). Algebraic Methods in Statistical Mechanics and Quantum Field Theory, Wiley-Interscience, ISBN 0-471-23900-3. Mathematically rigorous reference which provides extensive physics background.

G. Abrams, A. Louly, E. Pardo, C. Smith (2011). Flow invariants in the classi_cation of Leavitt path algebras. J. Algebra 333, pp. 202-231.

G. Abrams, G. Aranda Pino (2006). Purely in_nite simple Leavitt path algebras. J. Pure Appl. Algebra 207, pp. 553-563.

G. Abrams, G. Aranda Pino (2008). The Leavitt path algebras of arbitrary graphs. Houston J. Math. 34, no. 2, pp. 423-442.

G. Abrams, G. Aranda Pino (2006)., Purely infinite simple Leavitt path alge bras. J. Pure Appl. Algebra 207, pp. 553–563.

G. Aranda Pino, E. Pardo, M. Siles Molina (2009). Prime spectrum and primitive Leavitt path algebras. Indiana Univ. Math. J. 58, no. 2, pp. 869-890.

G. Bergman (1974). Coproduces and some universal ring constructions. Trans. Amer. Math. Soc. 200, pp. 33-88

P. Ara, K. Goodearl, E. Pardo (2002). K0 of purely in_nite simple regular rings. K-Theory 26, no. 1, pp. 69-100.

Sakai, S. (2011). C*-algebras and W*-algebras, Springer, ISBN 3-540-63633-1.

Segal, Irving (2007), "Irreducible representations of operator algebras", Bulletin of the American Mathematical Society **53** (2): pp. 73–88

T. Bates, D. Pask, I. Raeburn, W. Szymanski (2000). The C*-algebras of row-_nite graphs. New York J. Math. 6, pp. 307-324.

**Corresponding Author**

**Swati Maan***

M.Sc. (Mathematics), CSIR –JRF Qualified

**E-Mail –** arora.kips@gmail.com