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Introduction and Investigate a New Generalized Double Zeta Function

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Abstract – Within this section, we extant and learn a original generalized double zeta function $\zeta_{\lambda}^{\alpha,\beta,\gamma}$ which is a generalization of Hurwitz-Lerch zeta function and all its exacting cases. Amid course of our examination, we primary present its series definition and state some unique cases. Next, we make eight repeat relations, four differential formulae and three fractional derivatives including this function. shortly on, we build up one integral depiction, two series-integral representations and two double integral representations for this function. At extensive last, we obtain four series extensions for the function $\zeta_{\lambda}^{\alpha,\beta,\gamma}$. A few results got before by Bin Saad [7] and Jain [53] get after as unique instances of our primary discoveries.

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INTRODUCTION

The Generalized Riemann Zeta Function

The generalized Riemann zeta function is indicated by $\phi(y,z,a)$ beside with is distinct by [17, p.27, § 1.11, Eq.(1)], watch also [49, p.159, Eq.(5)]:

$$\phi(y,z,a)=\sum_{n=0}^{\infty}\frac{y^n}{(n+a)^z} \qquad (6.1.1)$$

$\left(a\in C/\{0,-1,-2,\ldots\},z\in C\right.$ when $|y|<1$ and Re(z)>1 when $\left.|y|=1\right)$

The over task is the generalization of the excellent generalized (Hurwitz) zeta function ζ (ν, a) and Riemann Zeta function ζ (ν) [17, p.24, § 1.10, Eq.(1) ; p.32, § 1.12,q.(1)].

Lin and Srivastava [81] identified and explore the successive generalization of the Hurwitz-Lerch zeta function $\phi(y,z,a)$ well-known as:

$$\phi_{\mu,\upsilon}^{(\rho,\sigma)}(y,z,a)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}}{(\upsilon)_{\sigma n}}\frac{y^n}{(n+a)^z} \qquad (6.1.2)$$

where $\mu\in C;a,\upsilon\in C/\{0,-1,-2,\ldots\};\rho,\sigma\in R^+;\rho<\sigma$ when $y,z\in C$; $\rho=\sigma$ for $y\in C$; $\rho=\sigma$, $z\in C$ for $|y|<1$; $\rho=\sigma$, $\text{Re}(z-\mu+\upsilon)>1$ for $|z|=1$.

If we take $\rho=\sigma=\upsilon=1$ in the over equation (6.1.2), we obtain

$$\phi_{\mu,1}^{(1,1)}(y,z,a)=\phi_{\mu}^{*}(y,z,a)\equiv\sum_{n=0}^{\infty}\frac{(\mu)_n y^n}{(n+a)^z n!} \qquad (6.1.3)$$

now $\phi_{\mu}^{*}(y,z,a)$ stands for generalized Hurwitz-Lerch zeta function distinct by Goyal and Laddha [35].

Jain [53, p.147, Eq.(6.2.5)] given and measured a innovative generalized Hurwitz-Lerch zeta function which is described and verbal to in the following way:

$$\phi_{\alpha,\beta,\gamma}(z,s,a)=\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\frac{z^n}{(n+a)^s} \qquad (6.1.4)$$

($\gamma,a\neq 0,-1,-2,\ldots s\in C$, when $|z|<1$ and $\text{Re}(s+\gamma-\alpha-\beta)>0$ when $|z|=1$).

For any complex number $a \neq 0$, Barnes double zeta function [5] is characterized by the following double series:

$$\zeta_2(z;a,\omega) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(a+n+\omega m)^{-z} \quad ; \operatorname{Re}(z)>2 \qquad (6.1.5)$$

Bin Saad [7] newly give a generalization of double zeta function as

$$\zeta_\lambda^\mu(x,y;z,a) = \sum_{m=0}^{\infty}(\mu)_m \phi(y,z,a+\lambda m)\frac{x^m}{m!} \qquad (6.1.6)$$

where
$|x|<1,\ |y|<1;\mu \in C/\{0,-1,-2,...\},\lambda \in C/\{0\};\ a \in C/\{-(n+\lambda m)\}$,
$\{n,\ m\} \in N \cup \{0\}$.

GENERALIZED DOUBLE ZETA FUNCTION

Roused by the different generalizations of Riemann zeta function, we introduce here another generalization of the double zeta function characterized by

$$\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a) = \sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!} \qquad (6.1.7)$$

or

$$\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{y^n}{(a+n+\lambda m)^z}\frac{x^m}{m!} \qquad (6.1.8)$$

$\{\alpha,\beta \in C, \gamma \in C/\{0,-1,-2,...\}$ for $|x|<1$ and $\operatorname{Re}(z+\gamma-\alpha-\beta)>0$ for $|x|=1$ $\lambda \in C/\{0\}$; $a \in C/\{-(n+\lambda m)\}, z \in C$ for $|y|<1$ and Re(z)>1 for $|y|=1\}$.

RELATIONSHIP OF $\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)$ WITH KNOWN FUNCTIONS:

(i) If we find y=0 and $\lambda = 1$ in (6.1.8), it yields the universal Hurwitz-Lerch zeta function $\phi_{\alpha,\beta,\gamma}(x,z,a)$.

$$\zeta_1^{\alpha,\beta,\gamma}(x;0,z,a) = \phi_{\alpha,\beta,\gamma}(x,z,a) \qquad (6.2.1)$$

(ii) additional on taking $\beta=\gamma=1$ in (6.2.1), we find

$$\zeta_1^{\alpha,1,1}(x;0,z,a) = \phi_\alpha^*(x,z,a) \qquad (6.2.2)$$

(iii) If we find x=0 in (6.1.8), we demonstrate at a generalization of the Hurwitz-Lerch zeta function $\phi(y,z,a)$ distinct by (6.1.1) as follows:

$$\zeta_\lambda^{\alpha,\beta,\gamma}(0;y,z,a) = \phi(y,z,a) \qquad (6.2.3)$$

(iv) For y=1, equation (6.2.3) yields the relation among generalized double

zeta function and Hurwitz (or generalized) zeta function as:

$$\zeta_\lambda^{\alpha,\beta,\gamma}(0;1,z,a) = \zeta(z,a) \qquad (6.2.4)$$

which, further for a =1, gives the affinity with the recognized Riemann zeta function $\zeta(z)$.

(v) It is not strong to observe from the definition (6.1.8), in link with (6.1.5) and (6.1.6) that

$$\zeta_\lambda^{\alpha,1,1}(x;y,z,a) = \zeta_\lambda^\alpha(x;y,z,a) \qquad (6.2.5)$$

and

$$\zeta_\lambda^{1,1,1}(1;1,z,a) = \zeta_2(z;a,\lambda) \qquad (6.2.6)$$

(I) RECURRENCE RELATIONS FOR THE FUNCTION $\zeta_\lambda^{\alpha,\beta,\gamma}$:

In this part, we shall get eight reappearance relations for the generalized double zeta function $\zeta_\lambda^{\alpha,\beta,\gamma}$.

If $\alpha,\beta \in C,\ \gamma \in C/\{0,-1,-2,...\}$ for $|x|<1$ then

(i) $\gamma\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)-(\gamma-\beta)\zeta_\lambda^{\alpha,\beta,\gamma+1}(x;y,z,a)=\beta\zeta_\lambda^{\alpha,\beta+1,\gamma+1}(x;y,z,a)$
(6.3.1)

(ii) $(\alpha-\beta)\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)=\alpha\zeta_\lambda^{\alpha+1,\beta,\gamma}(x;y,z,a)-\beta\zeta_\lambda^{\alpha,\beta+1,\gamma}(x;y,z,a)$
(6.3.2)

(iii) $(\alpha-\gamma+1)\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)=\alpha\zeta_\lambda^{\alpha+1,\beta,\gamma}(x;y,z,a)$ - $(\gamma-1)\zeta_\lambda^{\alpha,\beta,\gamma-1}(x;y,z,a)$ (6.3.3)

(iv) $\alpha\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)+(\beta-\gamma)x\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a+\lambda)-\alpha\zeta_\lambda^{\alpha+1,\beta,\gamma}(x;y,z,a)$

$+\alpha x\zeta_\lambda^{\alpha+1,\beta,\gamma}(x;y,z,a+\lambda)=-\gamma^{-1}(\gamma-\alpha)(\gamma-\beta)x\zeta_\lambda^{\alpha,\beta,\gamma+1}(x;y,z,a+\lambda)$
(6.3.4)

(v) $\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)-\zeta_\lambda^{\alpha-1,\beta,\gamma}(x;y,z,a)-x\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a+\lambda)$ = $-\gamma^{-1}(\gamma-\beta)x\zeta_\lambda^{\alpha,\beta,\gamma+1}(x;y,z,a+\lambda)$ (6.3.5)

(vi) $(2\alpha-\gamma)\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a)+(\beta-\alpha)x\zeta_\lambda^{\alpha,\beta,\gamma}(x;y,z,a+\lambda)$

$$=\alpha\zeta_{\lambda}^{\alpha+1,\beta,\gamma}(x;y,z,a)+(\alpha-\gamma)\zeta_{\lambda}^{\alpha-1,\beta,\gamma}(x;y,z,a)-\alpha x\zeta_{\lambda}^{\alpha+1,\beta,\gamma}(x;y,z,a+\lambda)$$
(6.3.6)

(vii) $\{\beta+x(\alpha-\beta)\}\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)-(\gamma-\beta)x\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a+\lambda)$

$$=\alpha x\left\{\zeta_{\lambda}^{\alpha+1,\beta,\gamma}(x;y,z,a)-\zeta_{\lambda}^{\alpha+1,\beta,\gamma}(x;y,z,a+\lambda)\right\}+(1-x)\beta\zeta_{\lambda}^{\alpha,\beta+1,\gamma}(x;y,z,a)$$

$$-\gamma^{-1}(\gamma-\alpha)(\gamma-\beta)x\zeta_{\lambda}^{\alpha,\beta,\gamma+1}(x;y,z,a+\lambda) \quad (6.3.7)$$

(viii) $\alpha\zeta_{\lambda}^{\alpha+1,\beta,\gamma}(x;y,z,a)-\alpha\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)=x\frac{d}{dx}\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a)$
(6.3.8)

Proof of (i): To prove (6.3.1), we first express the generalized double zeta function in its left hand side in series frame with the assistance of (6.1.7), we obtain

$$=\gamma\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}$$

$$-(\gamma-\beta)\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma+1)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}$$

$$=\gamma\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}\left[\frac{(\gamma+m)-(\gamma-\beta)}{(\gamma+m)}\right] \quad (6.3.9)$$

$$=\beta\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta+1)_m}{(\gamma+1)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}$$

$$=\beta\zeta_{\lambda}^{\alpha,\beta+1,\gamma+1}(x;y,z,a)$$

which is the right hand side of (6.3.1) and this completes the proof.

The proof of (6.3.2) and (6.3.3) can be simply developed in a parallel way.

Proof of (iv): To set up (6.3.4),we initiate with its left hand side and state the function $\zeta_{\lambda}^{\alpha,\beta,\gamma}$ in series form with the help of (6.1.7) in the first and third term, we obtain

$$=\alpha\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}+(\beta-\gamma)x\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a+\lambda)$$

$$-\alpha\sum_{m=0}^{\infty}\frac{(\alpha+1)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}$$
$$+\alpha x\zeta_{\lambda}^{\alpha+1,\beta,\gamma}(x;y,z,a+\lambda)$$

$$=\alpha\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}+(\beta-\gamma)x\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a+\lambda)$$

$$-(\alpha+m)\sum_{m=0}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{m!}$$
$$+\alpha x\zeta_{\lambda}^{\alpha+1,\beta,\gamma}(x;y,z,a+\lambda)$$

$$=-\sum_{m=1}^{\infty}\frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\phi(y,z,a+\lambda m)\frac{x^m}{(m-1)!}+(\beta-\gamma)x\zeta_{\lambda}^{\alpha,\beta,\gamma}(x;y,z,a+\lambda)$$

$$+\alpha x\zeta_{\lambda}^{\alpha+1,\beta,\gamma}(x;y,z,a+\lambda)$$

At last supplanting m by m+1 in the primary summation, we land at the correct hand side of (6.3.4) after a little improvement.

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