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An Analysis upon Ordinary Differential Equations and Its Applications

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Abstract – Differential equations is one of the oldest field in the modern mathematics. It is consider being the mathematics of applications in the fields of engineering, physics, life sciences and other areas in the field of mathematical modeling. Differential equations is a higher mathematics course of applications. This needs the prior knowledge of calculus as foundation for this course. This material is an introduction to ordinary differential equations. This covers the basic topics on ordinary differential equations that can serve as foundations in advanced studies in differential equations.

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INTRODUCTION

A differential equation is a mathematical equation that relates some function with its derivatives. In applications, the functions usually represent physical quantities, the derivatives represent their rates of change, and the equation defines a relationship between the two. Because such relations are extremely common, hence differential equations occur frequently in many branches of science including engineering, mathematics (both pure and applied mathematics), physics, economics, and biology. The subject of differential equations is built upon the subject of calculus. One possible explanation for this is to remember that a derivative describes a rate of change, so anytime it is used to describe how changes in one thing depend on changes in some other thing, differential equations are lurking in the background. Differential equations allow us to model changing patterns in both physical and mathematical problems.

Ordinary differential equations display the fleeting development of the significant variables by depicting their deterministic elements. The investigation of dynamical systems with ODEs is a develop field and in this manner, there is a rich writing committed to their examination and arrangement.

Tributes are utilized to display organic procedures on different levels running from quality articulation or flagging procedures on the cell level to the energy of medications all in all body level . Every one of these procedures have in like manner that their displaying with ODEs bears a significant level of uncertainty or potentially variability in both initial conditions and parameters. This is especially the situation when models are considered in a populace wide setting. At that point, uncertainty normally relates to boisterous estimations or the absence of information about individual systems, while variability alludes to varieties after some time in singular systems or inside the populace.

In light of the likelihood thickness capacity of the random initial values, the issue can be recast as a thickness spread issue. The advancement of the thickness work is depicted by a frst-arrange linear partial differential equation (PDE).

An ordinary differential equation can be composed in the form

$$\mathbf{L}(y) = f(x), \qquad (1)$$

Where $y(x)$ is an unidentified function. The equation is said to be *homogeneous* if $f(x) = 0$, giving then

$$\mathbf{L}(y) = 0 \qquad (2)$$

This is the most frequent usage for the term "homogeneous." The operator L is collected of a grouping of derivatives d/dx, d^2/dx^2, etc. The operator L is linear if

$$\mathbf{L}(y_1 + y_2) = \mathbf{L}(y_1) + \mathbf{L}(y_2), \qquad (3)$$

and

$$\mathbf{L}(\alpha y) = \alpha \mathbf{L}(y), \qquad (4)$$

where is a scalar. We can differentiate this definition of linearity with the definition of more general term "relative" given, which, while comparable, concedes a consistent inhomo-geneity.

For the rest of this investigation, we will take L to be a linear differential operator. The general form of L is

$$\mathrm{L} = P_N(x)\frac{d^N}{dx^N} + P_{N-1}(x)\frac{d^{N-1}}{dx^{N-1}} + \ldots + P_1(x)\frac{d}{dx} + P_0(x). \tag{5}$$

The ordinary differential equation, Eq. (1). is then linear when L has the form of Eq. (5).

Definition: The functions $y_1(x), y_2(x), \ldots, y_N(x)$ are said to be *linearly independent* when $C_1y_1(x) + C_2y_2(x) + \ldots + C_Ny_N(x) = 0$ is right only when $C_1 = C_2 = \ldots = C_N = 0$.

A homogeneous equation of order *N* can be shown to have *N* linearly independent solutions. These are called *complementary functions.* If $y_n\ (n = 1, \ldots, N)$ are the complementary functions of Eq. (2). then

$$y(x) = \sum_{n=1}^{N} C_n y_n(x), \tag{6}$$

is the general arrangement of the homogeneous Eq. (2). In dialect to be characterized in a future report, We can state the correlative functions are linearly independent and range the space of arrangements of the homogeneous equation; they are the bases of the invalid space of the differential operator L. If $y_p(x)$ is any *particular solution* of Eq. (1), the general solution to Eq. (2) is then

$$y(x) = y_p(x) + \sum_{n=1}^{N} C_n y_n(x). \tag{7}$$

Presently we might want to demonstrate that any arrangement $\phi(x)$ to the homogeneous equation $\mathrm{L}(y) = 0$ can be composed as a linear blend of the N correlative functions $y_n(x)$:

$$C_1y_1(x) + C_2y_2(x) + \ldots + C_Ny_N(x) \;=\; \phi(x). \tag{8}$$

We can form extra equations by taking a progression of subordinates up to $N-1$:

$$C_1y_1'(x) + C_2y_2'(x) + \ldots + C_Ny_N'(x) \;=\; \phi'(x), \tag{9}$$

$$\vdots$$

$$C_1y_1^{(N-1)}(x) + C_2y_2^{(N-1)}(x) + \ldots + C_Ny_N^{(N-1)}(x) \;=\; \phi^{(N-1)}(x). \tag{10}$$

This is a linear system of algebraic equations:

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_N \\ y_1' & y_2' & \cdots & y_N' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(N-1)} & y_2^{(N-1)} & \cdots & y_N^{(N-1)} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{pmatrix} = \begin{pmatrix} \phi(x) \\ \phi'(x) \\ \vdots \\ \phi^{(N-1)}(x) \end{pmatrix} \tag{11}$$

We could fathom Eq. (11) by Cramer's control, which requires the utilization of determinants. For a special arrangement, we require the determinant of the coefficient grid of Eq. (11) to be non¬zero. This specific determinant is known as the Wronskian W of $y_1(x), y_2(x), \ldots, y_N(x)$ and is characterized as

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_N \\ y_1' & y_2' & \cdots & y_N' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(N-1)} & y_2^{(N-1)} & \cdots & y_N^{(N-1)} \end{vmatrix}. \tag{12}$$

The condition $W \neq 0$ demonstrates linear autonomy of the functions $y_1(x), y_2(x), \ldots, y_N(x)$, since if $\phi(x) \equiv 0$, the main arrangement is $C_n = 0, n = 1, \ldots, N$. Tragically, the opposite isn't in every case genuine; that is, if $W = 0$, the correlative functions might be linearly reliant, however much of the time $W = 0$ without a doubt suggests linear reliance.

ORDINARY DIFFERENTIAL EQUATIONS WITH RANDOM INITIAL VALUES

In this section we show the scientific setting for ODEs with random initial values together with their subsequent arrangement.

We are keen on issues where the state $z \in \mathbb{R}^n$ of the system can be depicted by an ordinary differential equation of the form

$$\dot{z} = f(z\,|\,p)\,, \quad \text{with} \quad z(0) = z_0\,. \tag{13}$$

The correct hand side $f(\cdot|p) : \mathbb{R}^n \to \mathbb{R}^n$ may rely upon parameters $p \in \mathbb{R}^m$. Since we are occupied with an affectability examination regarding a model info comprising of both initial conditions z_0 and parameters p, we consider the broadened state variable $x := (z \;\; p)^T \in \mathbb{R}^d$, With $d = n + m$. This enables us to think about the impacts of varieties in z_0 and p all the while by setting

$$\dot{x} = F(x) := \begin{pmatrix} f(z\,|\,p) \\ 0 \end{pmatrix}, \quad \text{with} \quad x(0) = x_0 = \begin{pmatrix} z_0 \\ p \end{pmatrix}. \tag{14}$$

Let $|\cdot|$ signify a vector standard on $\mathbb{R}^d$ (e.g. the Euclidean standard). At that point, the accompanying theorem gives conditions for the presence and uniqueness of an answer $x(t),\, t \geq 0$

Theorem 1 (Existence Theorem of Picard-Lindelof). *Let F*

be locally Lipschitz continuous, i.e., there exists $L \geq 0$ *such that*

$$|F(x) - F(y)| \le L \cdot |x - y|, \qquad \forall x \in \mathbb{R}^d,\ y \in B_\kappa(x),$$

Where $B_\kappa(x) := \{y \in \mathbb{R}^d, |y - x|_2 \le \kappa\}$ signifies an open neighborhood around x. At that point, the initial value issue (14) has a one of a kind arrangement $x(t),\ t \ge 0$.

An adequate condition for neighborhood Lipschitz congruity is continuous differentiability of F as for the state variable x, which will be accepted from this time forward. Give us a chance to signify the development operator $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ with

$$\Phi_t x_0 := x(t)\,, \qquad (15)$$

which maps an initial state x_0 to its state at time t. The development operator has the accompanying properties:

(i) $\Phi_0 x = x$ for all $x \in \mathbb{R}^d$,

(ii) $\Phi_t(\Phi_{t'} x) = \Phi_{t+t'} x$ for all $x \in \mathbb{R}^d$ and $t, t' \in \mathbb{R}$,

(iii) $\Phi_t x$ is differentiate with respect to x for all $t \in \mathbb{R}$

reminder that by the first two properties, $\{\Phi_t\}_{t\in\mathbb{R}}$ forms a group, and therefore Φ_t is invertible with $\Phi_t^{-1} = \Phi_{-t}$

To scientifically differentiate the uncertainty or variability in initial values, we presume that $x_0 = X_0$ is a *random variable.* therefore, $\Phi_t x_0 = X_t$ is also a random variable and $\{X_t\}_{t\ge 0}$ a stochastic procedure. For any $t \ge 0$, let us denote with $u_t = u(t, \cdot),\ u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$,

the probability density function of the probability distribution of. X_t, i.e.

$$\mathbb{P}[X_t \le x] = \int_{-\infty}^{x} u_t(s)\ \mathrm{d}s \qquad (16)$$

The objective is to solve the following difficulty:

Problem 1 (Random Initial Value Problem). *Let the system be described by an ODE of the form*

$$\dot{x} = F(x)$$

Expect the initial value $x_0 = X_0$ is a random significant and has a known likelihood conveyance with thickness UQ. The issue is to process the likelihood thickness work ut related with the random state $x(t) = X_t$ on a limited interval $t \in [0, T]$

SOME APPLICATIONS

Is the subject of ordinary differential equations important? The ultimate answer to this question is certainly beyond the scope of this book. However, two main points of evidence for an affirmative answer are provided in this chapter:

- Ordinary differential equations arise naturally from the foundations of physical science.
- Ordinary differential equations are useful tools for solving physical problems.

You will have to decide if the evidence is sufficient. Warning: If you pay too much attention to philosophical issues concerning the value of a mathematical subject, then you might stop producing mathematics.

Origins of ODE: The Euler-Lagrange Equation-

Let us consider a smooth function $L : \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R} \to \mathbb{R}$, a pair of points $p_1, p_2 \in \mathbb{R}^k$, two real numbers $t_1 < t_2$, and the set $C := C(p_1, p_2, t_1, t_2)$ of all smooth curves $q : \mathbb{R} \to \mathbb{R}^k$ such that $q(t_1) = p_1$ and $q(t_2) = p_2$. Using this data, there is a function $\Phi : C \to \mathbb{R}$ given by

$$\Phi(q) = \int_{t_1}^{t_2} L(q(t), \dot{q}(t), t)\, dt. \qquad (17)$$

The Euler-Lagrange equation, an ordinary differential equation associated with the function *L*—called the *Lagrangian*—arises from the following problem: Find the extreme points of the function Φ. This variational problem is the basis for Lagrangian mechanics.

Recall from the calculus that an extreme point of a smooth function is simply a point at which its derivative vanishes. To use this definition directly for the function Φ, we would have to show that C is a manifold and that d> is differentiable. This can be done. However, we will bypass these requirements by redefining the notion of extreme point. In effect, we will define the concept of *directional derivative* for a scalar function on a space of curves. Then, an extreme point is defined to be a point where all directional derivatives vanish.

Recall our geometric interpretation of the derivative of a smooth function on a manifold: For a tangent vector at a point in the domain of the function, take a curve whose tangent at time *t* = 0 is the given vector, move the curve to the range of the function by composing it with the function and then differentiate the resulting curve at *t* = 0 to produce the tangent vector on the range that is the image of the original vector under the derivative of the function. In the context of the function Φ on the space of curves C, let us consider a curve $\gamma : \mathbb{R} \to C$. Note that for $s \in \mathbb{R}$, the point $\gamma(s) \in C$

is a curve $\gamma(s) : \mathbb{R} \to \mathbb{R}^k$ as defined above. So, in particular, if $t \in \mathbb{R}$, then $\gamma(s)(t) \in \mathbb{R}^k$. Rather than use the cumbersome notation $\gamma(s)(t)$, it is customary to interpret our curve of curves as a "variation of curves" in C, that is, as a smooth function $Q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^k$ with the end conditions"

$$Q(s,t_1) \equiv p_1, \qquad Q(s,t_2) \equiv p_2.$$

In this interpretation, $\gamma(s)(t) = Q(s,t)$.

Fix a point $q \in C$ and suppose that $\gamma(0) = q$, or equivalently that $Q(0,t) = q(t)$. Then, as *s* varies we obtain a family of curves called a *variation* of the curve *q.* The tangent vector to γ at *q* is, by definition, the curve $V : \mathbb{R} \to \mathbb{R}^k \times \mathbb{R}^k$ given by $t \mapsto (q(t), v(t))$ where

$$v(t) := \frac{\partial}{\partial s} Q(s,t)\Big|_{s=0}.$$

Of course, *v* is usually not in C because it does not satisfy the required end conditions. However, *v* does satisfy a perhaps different pair of end conditions, namely,

$$v(t_1) = \frac{\partial}{\partial s} Q(s,t_1)\Big|_{s=0} = 0, \qquad v(t_2) = \frac{\partial}{\partial s} Q(s,t_2)\Big|_{s=0} = 0.$$

Let us view the vector *V* as an element in the "tangent space of *C* at *q."*

What is the directional derivative $D\Phi(q)V$ of Φ at *q* in the direction *V?* Following the prescription given above, we have the definition

$$\begin{aligned} D\Phi(q)V &:= \frac{\partial}{\partial s}\Phi(Q(s,t))\Big|_{s=0} \\ &= \int_{t_1}^{t_2} \frac{\partial}{\partial s} L(Q(s,t), \frac{\partial}{\partial t} Q(s,t), t)\Big|_{s=0} dt \\ &= \int_{t_1}^{t_2} \Big(\frac{\partial L}{\partial q}\frac{\partial Q}{\partial s} + \frac{\partial L}{\partial \dot q}\frac{\partial^2 Q}{\partial s \partial t}\Big)\Big|_{s=0} dt. \end{aligned}$$

After evaluation at $s = 0$ and an integration by parts, we can rewrite the last integral to obtain

$$\begin{aligned} D\Phi(q)V &= \int_{t_1}^{t_2} \Big[\frac{\partial L}{\partial q}(q(t),\dot q(t),t) - \frac{d}{dt}\Big(\frac{\partial L}{\partial \dot q}(q(t),\dot q(t),t)\Big)\Big] \frac{\partial Q}{\partial s}(0,t)\, dt \\ &= \int_{t_1}^{t_2} \Big[\frac{\partial L}{\partial q}(q(t),\dot q(t),t) - \frac{\partial}{\partial t}\Big(\frac{\partial L}{\partial \dot q}(q(t),\dot q(t),t)\Big)\Big] v(t)\, dt. \end{aligned} \qquad (18)$$

If $D\Phi(q)V = 0$ for all vectors *V,* then the curve *q* is called an *extremal.* In other words, *q* is an extremal if the last integral in equation (18) vanishes for all smooth functions *v* that vanish at the points t_1 and t_2.

Origins of ODE: Classical Physics-

The fundamental laws of all of classical physics can be reduced to a few formulas! For example, a complete theory of electromagnetics is given by Maxwell's laws

$$\begin{aligned} \operatorname{div} \mathbf{E} &= \rho/\epsilon_0, \\ \operatorname{curl} \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\ \operatorname{div} \mathbf{B} &= 0, \\ c^2 \operatorname{curl} \mathbf{B} &= \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t} \end{aligned}$$

and the conservation of charge

$$\operatorname{div}(\mathbf{j}) = -\frac{\partial \rho}{\partial t}.$$

Here E is the called electric field, B is the magnetic field, ρ is the charge density, j is the current, ϵ_0 is a constant, and *c* is the speed of light. The fundamental law of motion is Newton's law

$$\frac{d\mathbf{p}}{dt} = F$$

"the rate of change of the momentum is equal to the sum of the forces." The (relativistic) momentum of a particle is given by

$$\mathbf{p} := \frac{m}{\sqrt{1 - v^2/c^2}}\mathbf{v}$$

where, as is usual in the physics literature, $v := |\mathbf{v}|$ and the norm is the Euclidean norm. For a classical particle (velocity much less than the speed of light), the momentum is approximated by p = mv. There are two fundamental forces: The gravitational force

$$F = -\frac{GMm}{r^2}\mathbf{e}_r$$

on a particle of mass *m* due to a second mass *M* where *G* is the universal gravitational constant and e_r is the unit vector at *M* pointing in the direction of *m:* and the Lorentz force

$$F = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where *q* is the charge on a particle in an electromagnetic field. That's it!

The laws of classical physics seem simple enough. Why then is physics, not to mention engineering, so complicated? Of course, the answer is that in almost all real world applications there are *lots* of particles and the fundamental laws act all at once. When we *try* to isolate some experiment or some physical phenomenon from all the other stuff in the universe, then we are led to develop "constitutive" laws that approximate the true situation. The equations of motion then contain many additional "forces." Let us consider a familiar example. When we model the motion of a spring, we use Hooke's force law to obtain the equation of motion in the form

$$m\ddot{x} = -\omega_0 x.$$

However, Hooke's force law is not one of the two fundamental force laws. In reality, the particles that constitute the spring obey the electromagnetic force law and the law of universal gravitation. However, if we attempted to model the fundamental forces acting on each of these particles, then the equations of motion would be so complex that we would not be able to derive useful predictions.

What law of nature are we using when we add viscous damping to a Hookian spring to obtain the differential equation

$$m\ddot{x} = -\alpha\dot{x} - \omega_0 x$$

as a model? The damping term is supposed to model a force due to friction. But what is friction? There are only two fundamental forces in classical physics and only four known forces in modern physics. Friction is not a nuclear force and it is not due to gravitation. Thus, at a fundamental level it must be a manifestation of electromagnetism. Is it possible to derive the linear form of viscous damping from Maxwell's laws? This discussion could become very philosophical!

The important point for us is an appreciation that the law of motion—so basic for our understanding of the way the universe works is expressed as an ordinary differential equation. Newton's law, the classical force laws, and the constitutive laws are *the* origin of ordinary differential equations. Apparently, as Newton said. "Solving differential equations is useful."

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