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Equicontinuity and Bilinear Mappings

Santosh Kumar[1]* Dr. Raj Kumar[2] Dr. Sumit Kumar[3]

[1] Research Scholar, P.G Department of Mathematics, Patna University, Patna, Bihar

[2] St Xavier's College of Management and Technology, Digha Ashiyana Road, Patna, Bihar

[3] Assistant Professor, Zeal College of Engineering and Research

Abstract – We consider various types of boundedness, equicontinuity and sequential equicontinuity for sets of separately continuous bilinear mappings between topological modules. Our purpose here is to establish relations among the various notions of boundedness (resp. equicontinuity, sequential equicontinuity) under consideration, as well as to establish relations among notions of a different nature; for example, to obtain conditions under which point wise boundedness implies separate equicontinuity.

Keywords: Bilinear Mappings, Equicontinuity

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INTRODUCTION

A denotes a commutative topological ring with an identity element and A∗ denotes the multiplicative group of all invertible elements of A. All modules under consideration are unitary A-modules. E, F and G represent topological A-modules, M (resp. N) represents a set of bounded subsets of E (resp. F), and L_{sep}(E,F; G) represents the A-module of all separately continuous A-bilinear mappings from E × F into G.

Definition

Let X ⊂ L_{sep}(E,F; G).

(B1) X is point wise bounded if, for each (x, y) ∈ E × F, the set

$$\mathfrak{X}(x,y) = \{u(x,y); u \in \mathfrak{X}\}$$

Is bounded in G.

(B2)(a) X is M-uniformly bounded if, for each y ∈ F and for each B ∈ M, the set

$$\{u(x,y); u \in \mathfrak{X}, x \in B\}$$

Is bounded in G.

(B2)(b) X is N -uniformly bounded if, for each x ∈ E and for each C ∈ N , the set

$$\{u(x,y); u \in \mathfrak{X}, y \in C\}$$

Is bounded in G.

(B3) X is (M, N)-uniformly bounded if X is M-uniformly bounded and N -uniformly bounded.

(B4) X is (M×N)-uniformly bounded if, for each B ∈ M and for each C ∈ N , the set

$$\mathfrak{X}(B \times C) = \{u(x,y); u \in \mathfrak{X}, x \in B, y \in C\}$$

Is bounded in G.

Remark

Since every subset of a topological module over a discrete ring is necessarily bounded, there is no interest in the study of the notions just defined when A is a discrete ring.

Remark

(a) If $\bigcup_{B\in\mathcal{M}} B = E$ (resp. $\bigcup_{C\in\mathcal{N}} C = F$) then (B2)(a) implies (B1) (resp. (B2)(b) implies (B1)).

(b) If $\bigcup_{C\in\mathcal{N}} C = F$ (resp. $\bigcup_{B\in\mathcal{M}} B = E$), then (B4) implies (B2)(a) (resp. (B4) implies (B2)(b)). In particular, if $\bigcup_{B\in\mathcal{M}} B = E$ and $\bigcup_{C\in\mathcal{N}} C = F$, then (B4) implies (B3).

There are examples showing that the reverse implications in Remark are not valid in general; see

[2]. In the sequel we shall see conditions under which such reverse implications hold.

Definition

Let X ⊂ L_{sep}(E,F; G).

(E1)(a) X is left equicontinuous if, for each y ∈ F, the set

$$\{x \in E \mapsto u(x, y) \in G; u \in \mathfrak{X}\}$$

Is equicontinuous

(E1)(b) X is right equicontinuous if, for each x ∈ E, the set.

$$\{y \in F \mapsto u(x, y) \in G; u \in \mathfrak{X}\}$$

Is equicontinuous.

(E2)(a) X is N -equihypocontinuous if, for each C ∈ N , the set

$$\{x \in E \mapsto u(x, y) \in G; u \in \mathfrak{X}, y \in C\}$$

Is equicontinuous.

(E2)(b) X is M-equihypocontinuous if, for each B ∈ M, the set

$$\{y \in F \mapsto u(x, y) \in G; u \in \mathfrak{X}, x \in B\}$$

Is equicontinuous.

(E3) X is (M, N)-equihypocontinuous if X is M-equihypocontinuous and N -equihypocontinuous.

(E4) X is equicontinuous if, for each (x, y) ∈ E×F and for each neighborhood W of zero in G, there exist a neighborhood U of zero in E and a neighborhood V of zero in F such that the relations u ∈ X, x0 ∈ U, y0 ∈ V imply u(x0 +x, y0 +y)−u(x, y) ∈ W.

Remark

If $\bigcup_{C\in\mathcal{N}} C = F$ (resp. $\bigcup_{B\in\mathcal{M}} B = E$), then (E2)(a) implies (E1)(a) (resp. (E2)(b) implies (E1)(b)). In particular, if $\bigcup_{B\in\mathcal{M}} B = E$ and $\bigcup_{C\in\mathcal{N}} C = F$, then (E3) implies (E1)(a) and (E1)(b).

Remark

Suppose that the product of any neighborhood of zero in A by any neighborhood of zero in E (resp. F) is a neighborhood of zero in E (resp. F). Then (E4) implies (E2)(a) (resp. (E4) implies (E2)(b)); in particular, if these two properties hold, then (E4) implies (E3). In fact, let X ⊂ Lsep(E,F; G) be equicontinuous, let C ∈ N , and assume that the product of any neighborhood of zero in A by any neighborhood of zero in E is a neighborhood of zero in E. Given an arbitrary neighborhood W of zero in G, there are a neighborhood U of zero in E and a neighborhood V of zero in F such that X(U × V) ⊂ W. By the boundedness of C, there exists a neighborhood L of zero in A such that LC ⊂ V . Thus

$$\mathfrak{X}((LU) \times C) = \mathfrak{X}(U \times (LC)) \subset \mathfrak{X}(U \times V) \subset W.$$

Since, by assumption, LU is a neighborhood of zero in E, we have just verified that the set

$$\{x \in E \mapsto u(x, y) \in G; u \in \mathfrak{X}, y \in C\}$$

Is equicontinuous. Therefore X is N -equihypocontinuous. By interchanging the roles of E and F, we conclude that the other assertion is also true. There are examples showing that the reverse implications in Remarks are not valid in general; see [2]. In the sequel we shall see conditions under which such reverse implications hold.

BILINEAR MAPPINGS

The idea of a together consistent bilinear mapping between topological vectors spaces has been considered broadly; for instance, [13, 14, 15] for more data. Specifically, when we manage the normed spaces structure, these mappings convey limited sets (concerning the item topology) to limited sets. Then again, tensor items are a productive and helpful device in changing over a bilinear mapping to a direct administrator in any setting; for instance, the projective tensor item for normed spaces and the Fremlin projective tensor items for vector grids and Banach cross sections [8, 9]. In a topological vector space setting, we can consider two distinctive non-identical approaches to characterize a limited bilinear mapping. For reasons unknown, these parts of boundedness are as it were "middle of the road" thoughts of a mutually persistent one. Then again, various sorts of limited straight administrators between topological vector spaces and a portion of their properties have been examined [16, 18]. In this area, by utilizing the idea of projective tensor item between locally raised spaces, we show that, it could be said, various thoughts of a limited bilinear mapping match with various parts of a limited administrator. We demonstrate that for two limited straight administrators, the tensor item administrator likewise has a similar boundedness property, also.

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Definition

Let X, Y , and Z be topological vector spaces. A bilinear mapping σ : X × Y → Z is said to be:

(i) n-bounded if there exist some zero neighborhoods U ⊆ X and V ⊆ Y such that σ (U × V) is bounded in Z;

(ii) b-bounded if for any bounded sets B1 ⊆ X and B2 ⊆ Y , σ(B1 × B2) is bounded in Z We first show that these concepts of bounded bilinear mappings are not equivalent.

Example

Let X = R N be the space of every genuine arrangement with the Tychonoff item topology. Consider the bilinear mapping σ : X × X → X characterized by σ(x, y) = xy where x = (xi), y = (yi) besides, the thing is point shrewd. It is successfully affirmed that σ is b-constrained; anyway since X isn't secretly restricted, it can't be a n-restricted bilinear mapping.

It isn't difficult to see that every n-constrained bilinear mapping is commonly incessant and each together relentless bilinear mapping is b-restricted, with the objective that these thoughts of constrained bilinear mappings are related to together steady bilinear mappings. Note that a b-constrained bilinear mapping needs not be commonly steady, even autonomously tenacious; by a freely incessant bilinear mapping; we mean one which is relentless in all of its portions. Consider the going with model

Example

Let X be the space C[0, 1], consisting of all real continuous functions on [0, 1]. Suppose τ1 is the topology generated by the seminorms px(f) = |f(x)|, for each x ∈ [0, 1], and τ_2 is the topology induced by the metric defined via the formulae

$$d(f,g) = \int_0^1 \frac{|f(x)-g(x)|}{1+|f(x)-g(x)|}dx.$$

Consider the bilinear mapping σ : (X, τ_1)×(X, τ_1) → (X, τ_2) defined by σ(f, g) = fg. It is easy to show that σ is a b-bounded bilinear mapping. But is it not even separately continuous; for example the mapping g = $1_{X,}$ the identity operator from (X, τ_1) into (X, τ_2), is not continuous. To see this, suppose

$$V = \{f \in X : d(f,0) < \frac{1}{2}\}.$$

V is a zero neighborhood in (X, τ_2). If the identity operator is continuous, there should be a zero neighborhood U ⊆ (X, τ_1) with U ⊆ V . Therefore, there are {x_1, . . . , x_n} in [0, 1] and ε > 0 such that

$$U = \{f \in X, |f(x_i)| < \varepsilon, i = 1, \dots, n\}.$$

For each subinterval [x_i , x_{i+1}], consider positive reals αi and α_{i+1} such that $x_i < \alpha_i < \alpha_{i+1} < x_{i+1}$. For an n ∈ N, Define,

$$f_i(x) = \begin{cases} \frac{n(x-x_i)}{\alpha_i - x_i}, & \text{if } x_i \le x \le \alpha_i, \\ n, & \text{if } \alpha_i \le x \le \alpha_{i+1}, \\ \frac{n(x-x_{i+1})}{\alpha_{i+1}-x_{i+1}}, & \text{if } \alpha_{i+1} \le x \le x_{i+1}. \end{cases}$$

Now consider the continuous function f on [0, 1] defined by $f_i's$ obviously $f \in U$. Put $\beta = \sum_{i=1}^n(\alpha_{i+1}-\alpha_i)$ We can choose n ∈ N and β in such a way that $\frac{\beta n}{n+1} > \frac{1}{2}$. Thus,

$$\int_0^1 \frac{|f(x)|}{1+|f(x)|}dx > \frac{\beta n}{n+1} > \frac{1}{2}.$$

This finishes the case. In what pursues, by utilizing the idea of the projective tensor result of locally arched spaces, we are demonstrating that these ideas of limited bilinear mappings are, truth be told, the various sorts of limited administrators characterized on a locally curved topological vector space. Review that if U and V are zero neighborhoods for locally raised spaces X and Y, separately, at that point co(U ⊗ V) is a run of the mill zero neighborhood for the locally curved space X⊗πY .

Proposition

Let X, Y and Z be locally convex vector spaces and θ : X × Y → X⊗πY be the canonical bilinear mapping. If ϕ : X × Y → Z is an nbounded bilinear mapping, there exists an nb-bounded operator T : X⊗πY → Z such that T ◦ θ = ϕ.

Proof

By [15], there is a straight mapping T : X⊗πY → Z with the end goal that T ◦ θ = ϕ. Subsequently, it is sufficient to show that T is nb-limited. Since ϕ is nbounded, there are zero neighborhoods U ⊆ X and V ⊆ Y with the end goal that ϕ(U × V) is limited in Z. Give W a chance to be a subjective zero neighborhood in Z. There is r > 0 with ϕ(U × V) ⊆ rW. It isn't difficult to show that T (U ⊗ V) ⊆ rW, so T (co(U ⊗ V)) ⊆ rW. However, by the reality referenced before this recommendation, co(U ⊗V) is a zero neighborhood in the space X⊗πY . This finishes the verification

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Proposition

Let X, Y and Z be locally convex vector spaces and θ : X × Y → X⊗πY be the canonical bilinear mapping. If ϕ : X ×Y → Z is a b-bounded bilinear mapping, there exists a bb-bounded operator T : X⊗πY → Z such that T ∘ θ = ϕ.

Proof

As in the proof of the previous theorem, the existence of the linear mapping T : X⊗πY → Z such that T ∘ θ = ϕ follows by [15]. We prove that the linear mapping T is bb-bounded. Consider a bounded set B ⊆ X⊗πY . There exist bounded sets B_1⊆ X and B_2⊆ Y such that B ⊆ B_1⊗ B_2. To see this, put

$$B_1 = \{x \in X, \exists\, y \in Y, \text{ such that } x \otimes y \in B\},$$
$$B_2 = \{y \in Y, \exists\, x \in X, \text{ such that } x \otimes y \in B\}.$$

It is not difficult to see that B_1 and B_2 are bounded in X and Y , respectively, and B ⊆ B_1⊗ B_2. Also, since θ is jointly continuous, B_1⊗ B_2 is also bounded in X⊗πY . Thus, from the inclusion

$$T(B) \subseteq T(B_1 \otimes B_2) = T \circ \theta(B_1 \times B_2) = \varphi(B_1 \times B_2)$$

And using the fact that ϕ is a b-bounded bilinear mapping, it follows that T is a bb-bounded linear operator. This concludes the claim and completes the proof of the proposition

Remark

Note that the similar result for jointly continuous bilinear mappings between locally convex spaces is known and commonly can be found in the contexts concerning topological vector spaces [15]. We are going now to investigate whether or not the tensor product of two operators preserves different kinds of bounded operators between topological vector spaces. The response is affirmative. Recall that for vector spaces X, Y , Z, and W, and linear operators T : X → Y , S : Z → W, by the tensor product of T and S, we mean the unique linear operator T ⊗S : X ⊗Z → Y ⊗W defined via the formulae

$$(T \otimes S)(x \otimes z) = T(x) \otimes S(z);$$

One may consult [14] for a comprehensive study regarding the tensor product operators.

Theorem

Let X, Y , Z, and W be locally convex spaces, and T : X → Y and S : Z → W be nb-bounded linear operators. Then the tensor product operator T ⊗ S : X⊗πZ → Y ⊗πW is nb-bounded.

Proof

Let U ⊆ X and V ⊆ Z be two zero neighborhoods such that T (U) and S(V) are bounded subsets of Y and W, respectively. Let O1 ⊆ Y and O_2⊆ W be two arbitrary zero neighborhoods. There exist positive reals α and β with T (U) ⊆ αO_1 and S(V) ⊆ βO_2. Then

$$(T \otimes S)(U \otimes V) = T(U) \otimes S(V) \subseteq \alpha\beta(O_1 \otimes O_2) \subseteq \alpha\beta \mathrm{co}(O_1 \otimes O_2),$$

So that

$$(T \otimes S)(\mathrm{co}(U \otimes V)) \subseteq \alpha\beta \mathrm{co}(O_1 \otimes O_2)$$

This is the desired result.

Theorem

Suppose X, Y , Z, and W are locally convex spaces, and T : X → Y and S : Z → W are bb-bounded linear operators. Then the tensor product operator T ⊗ S : X⊗πZ → Y ⊗πW is also bb-bounded

Proof

Fix a bounded set B ⊆ X⊗πZ. By the argument used in Proposition, there are bounded sets B1 ⊆ X and B_2⊆ Z with B ⊆ B_1⊗B_2. Let O_1⊆ Y and O_2⊆ W be two arbitrary zero neighborhoods. There are positive reals α and β such that T (B_1) ⊆ αO_1 and S(B_2) ⊆ βO_2. Therefore,

$$(T\otimes S)(B) \subseteq (T\otimes S)(B_1\otimes B_2) = T(B_1)\otimes S(B_2) \subseteq \alpha\beta(O_1\otimes O_2) \subseteq \alpha\beta \mathrm{co}(O_1\otimes O_2),$$

Hence

$$(T \otimes S)(B) \subseteq \alpha\beta \mathrm{co}(O_1 \otimes O_2),$$

As required

Theorem

Suppose X, Y , Z, and W are locally convex spaces, and T : X → Y and S : Z → W are continuous linear operators. Then the tensor product operator T ⊗ S : X⊗πZ → Y ⊗πW is jointly continuous.

Proof

Let O_1⊆ Y and O_2⊆ Z be two arbitrary zero neighborhoods. There exist zero neighborhoods U ⊆ X and V ⊆ Z such that T (U) ⊆ O_1 and S(V) ⊆ O_2. It follows

$$(T \otimes S)(U \otimes V) = T(U) \otimes S(V) \subseteq (O_1 \otimes O_2) \subseteq \mathrm{co}(O_1 \otimes O_2),$$

So that

$$(T \otimes S)(\mathrm{co}(U \otimes V)) \subseteq \mathrm{co}(O_1 \otimes O_2),$$

As claimed.

CONCLUSION

It should be mentioned that this paper was written under the influence of where the same notions for sets of separately continuous bilinear mappings between topological vector spaces have been studied. Throughout this work, A denotes a commutative topological ring with an identity element and A∗ denotes the multiplicative group of all invertible elements of A. All modules under consideration are unitary A-modules. E, F and G represent topological A-modules, M (resp. N) represents a set of bounded subsets of E (resp. F), and Lsep(E,F; G) represents the A-module of all separately continuous A-bilinear mappings from E × F into G.

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Corresponding Author

Santosh Kumar*

Research Scholar, P.G Department of Mathematics, Patna University, Patna, Bihar

santoshrathore.kumar20@gmail.com