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Analytic Solutions of a Second-Order Functional Differential Equation

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Abstract – In this paper we examine the nature of the secondary order differential equation analytical solutions with a state derivative formal delay $\ddot{x}(z)=x(p(z)+b\dot{x}(z))$. ***Considering a convergent power series g(z) of a secondary equation*** $\gamma^2\ddot{g}(\gamma z)\dot{g}z=[g(\dot{y}z)-p(g(\gamma z))]\,\gamma\dot{g}(\gamma z)(\dot{g}(z))^2+\ddot{p}(g(z))(\dot{g}(z))^3+\gamma\dot{g}(\gamma z)\ddot{g}(z)$ ***with the relation p(z) +*** $b\dot{x}(z)=g(\gamma g^{-1}(z))$, ***we obtain an analytic solution x(z). Furthermore, we characterize a polynomial solution when p(z) is a polynomial. We built a corresponding auxiliary equation with parameter to obtain analytical solutions of the problem.*** γ. ***The existence of solutions of an auxillary equation depends on the condition of a parameter*** γ, ***such as*** γ ***is in the unit circle and*** γ ***is a root of unity which satisfies the Diophantine condition.***

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1. INTRODUCTION:

The functional differential equation,

$$x^{(m)}(z)=f(z,x^{(m_1)}(z-\tau_1(z)),x^{(m^2)}(z-t_2(z)),\ldots,\,{}^{(mk)}(z-\tau_k(z))), \qquad (1)$$

where all $m_i\geq 0,\tau_i\geq 0$, provide a physical or biological system statistical model, in which the rate of shift in a system is not only dictated by its current condition, but also by its past (see [1, 2]). Many scholars have researched the nature and uniqueness of a number of these equations in recent years. In 1984, in the Banch fixed point theorem, Eder [3] classified solutions for operational differential equations.

In 2001[7], Si and Wang studied the nature of a second-order functional differential equation analytical solution:

$$\ddot{x}(z)=x(p(z)+b\dot{x}(z)) \qquad (2)$$

In 2009, Liu and Li [8] studied the equation

$$\ddot{x}(z)=\dot{x}(z)=x(az+bx(z)) \qquad (3)$$

Observe that (3) can be reduced to (2) by setting c = 0.

Next, the equation

$$\ddot{x}(z)=(az+b\dot{x}(z)) \qquad (4)$$

has been studied by Si and Wang [9].

In order to obtain analytic solutions of (4), they constructed a corresponding auxiliary equation with parameter γ. The presence of auxiliary equation solutions depends on the parameter state γ, such as γ is in the circle of units and γ The source of peace fulfilling the state of the Diophantine. In this paper we examine the nature of the secondary order differential equation analytical solutions with a state derivative formal delay.

$$\ddot{x}(z)=(p(z)+b\dot{x}(z)) \qquad (5)$$

If (z) = az, then (5) reduces to (4)

Please note that three cases of parameters γ – bis - would be studied in the relevant auxiliary equation in this article. One is that TER γ is the origin of unity that satisfies the requirement of Brjuno. To construct an Equation supplementary, we set

$$y(z)=(z)+b\dot{x}(z) \qquad (6)$$

Then

$$x(z)=(z0)+\frac{1}{b}\int_{z_0}^{z}(y(s)-p(s))ds \qquad (7)$$

Where Z_0 is a continuous complex, In particular, we have

$$x((z)) = x(z_0) + \frac{1}{b}\int_{z_0}^{y(z)} (y(s) - p(s))ds \qquad (8)$$

Applying relations (6) and (8) to (5), we obtain

$$\frac{1}{b}(\dot{y}(z) - \dot{p}(z)) = (z_0) + \frac{1}{b}\int_{z_0}^{y(z)} (y(s) - p(s))ds \qquad (9)$$

We construct the corresponding equation by differentiating both sides of (9) with respect to z. These yields.

$$\ddot{y}(z) - \ddot{p}(z) = (y(y(z)) - p(y(z)))\dot{y}(z) \qquad (10)$$

2. POLYNOMIAL SOLUTIONS OF (5)

In this section, we let (z) be a polynomial. Then, The polynomial solution of our research (5)

Have the secondary equation

$$y^2\ddot{g}(\gamma z)gz = [g(\gamma z) - p(g(\gamma z))]\gamma \dot{g}(\gamma z)(\dot{g}(z))^2 + \ddot{p}(g(z))(\dot{g}(z))^3 + \gamma \dot{g}(\gamma z)\ddot{g}(z) \qquad (11)$$

With the relation $p(z) + b\dot{x}(z) = g(\gamma g^{-1}(z)),$ we obtain an analytic solution x(z). Furthermore, we characterize a polynomial solution when p(z) is a polynomial.

Theorem 1 : For a polynomial (z), the equation

$$\ddot{x}(z) = (p(z) + b\dot{x}(z)) \qquad (12)$$

has a nontrivial polynomial solution if and only if p(z) = p_0

with $p_0 \neq 0$ or $p(z) \equiv p_0 + p_1 z$ with $p_1 \neq 0$

Proof

Necessity. Assume that $(z) = \Sigma_{k=0}^{n} x_k z^k$ is a nontrivial polynomial solution of (12). Le $(z) = \Sigma_{k=0}^{n} p_k z^k$ with $p_m \neq 0$.

Observe that $\ddot{x}(z) = 0$ when n = 0. This implies $(z) \equiv 0$. From now on, we let $n \geq 1$.

We consider 3 cases.

⇒ **Case 1**(m = 0). That is, $(z) = p_0 \neq 0$. Equation (12) becomes

$$2x_2 + 6x_3 z + \ldots + n(n-1)x_n z^{n-2} = x_0 + x_1(q(z)) + \ldots.. + x_n(q(z))^n \qquad (13)$$

Where

$$q(z) = (p_0 + bx_1) + 2bx_2 z + \ldots. + nbx_n z^{n-1}$$

If n = 1, then (13) changes to 0 = x0 + xi (p0 + bx1).

Next, we consider n > 2.

Comparing coefficient of zn$^{(n-1)}$ in (13), we have x_n = 0.

Equation (13) is reduced to

$$2x_2 + \ldots. + (n-1)(n-2)x_{n-1}z^{n-3} = x_0 + x_1(q(z)) + \ldots + x_{n-1}(qz)) \qquad (14)$$

Where

$q(z) = (p_0 + bx_1) + 2bx_2 z + \ldots. + (n-1)bx_{n-1}z^{n-2}$.

Comparing coefficient of $z^{(n-1)(n-2)}$ in (14), we have $x_{n-1} = 0$. Then, repeating the above method, we obtain $x_{n-2} = \ldots.. = x_2 = 0$ and also have $0 = x = x_0 + x_1(p_0 + bx_1)$.

By choosing an arbitrary nonzero x1, say η, both situations yield a nontrivial solution $x(z) = -\eta(p_0 + b\eta) + \eta z$.

⇒ **Case 2** (m = 1). That is, (z) = p_0 + p_iz, where $p_1 \neq 0$.

Equation (12) becomes

$$2x_2 + 6x_3 z + \ldots.. + n(n-1)x_n z^{n-2} = x_0 + x_1(q(z)) + \ldots.. + x(q(z)) \qquad (15)$$

Where

$$(z) = (p_0 + bx_1) + (p_1 + 2bx_2)z + 3bx_1 z^2 + \ldots. + nbx_n z^{n-1}$$

By comparing coefficient of constant term and z in (15), we obtain $(z) \equiv 0 \text{ for } n = 1$.

Next, we consider n ≥ 2.

Comparing coefficient of $z^{n(n-1)}$ in (15), we have $x_n(nbx_n)^n = 0$, which implies x_n = 0. Therefore, (15) is reduced to

$$2x_2 + 6x_3 z + \ldots. + (n-1)(n-2)x_{n-1}z^{n-3} = x_0 + x_1(q(z)) + \ldots.. + x_{n-1}(q(z))^{n-1} \qquad (16)$$

Where

$q(z) = (p_0 + bx_1) + (p_1 + 2bx_2)z + 3bx_3 z^2 + \ldots + (n-1)$

$bx_{n-1}z^{n-2}$

Comparing coefficient of $z^{(n-1)(n-2)}$ in (16), we have $x_{n-1} = 0$. Continuing this process, we obtain $2x_2 = x_0 + x_1(q(z)) + x_2(q(z))^2$, where $q(z) = (p_0 + bx_1) + (p_1 + 2bx_2)z$.

By comparing coefficient of z2 together with $x_2 \neq 0$, we obtain a nontrivial solution.

$(z) = (-p_1/b) - \eta(p_0 + b\eta) + (p_1/2b)(p_0 + b\eta)^2 + \eta z - (p_1/2b)z^2$, where η is an arbitrary constant.

Note that if $z_2 = 0$ then $(z) \equiv 0$

$\Rightarrow$ **Case 3** ($m \geq 2$).

We consider 2 subcases.

Subcase 3.1 ($m < n - 1$). Equation (12) becomes

$$2x_2 + 6x_3 z + \dots + n(n-1)x_n z^{n-2} = x_0 + x_1(q(z)) + \dots + x_n(q(z))^n \tag{17}$$

Where

$q(z) = p_0) + bx_1) + \dots + (p_m + (m+1)bx_{m-1})$ $z^m + (m+2)bx_{m+2}z^{m+1} + \dots + nbx_n z^{n-1}$. By using method of undetermined coefficient, we obtain $x_n = \dots = x_{m,2} = 0$.

Consequently, (46) is reduced to

$$2x_2 + 6x_3 z + \dots + (m+1)(m)x_{m+1} z^{m-1} = x_0 + x_1(q(z)) + \dots + x_{m+1}(q(z))^{m+1} \tag{18}$$

Where

$q(z = (p_0 + bx_i) + \dots + (pm + (m+1)bx_{m+1})z^m$

Comparing coefficient of $z^{(m)(m+1)}$ in (18), We have x_{m+1} $(p_m + (m+1)\, bx_{m+1})^{m+1} = 0$. If $x_{m+1} = 0$, then $x_m = \dots = x_0 = 0$.

That is $(z) \equiv 0$ which is a contradiction, Therefore,

$P_m + (m + 1)\, x_{m+1} = 0$. Substituting this relation in (17) and repeating this process, we get $x_{m-k} = -p_{m-(k+1)} / (m - k)\, b$ for $k = -1 \dots m-2$. Using this fact in (18) and then comparing the coefficient of z^t ($l = 1, \dots, m - 1$), we obtain $x_2 = \dots = x_{m+1} = 0$. This yield $p = 0$, which is a contradiction.

Subcase 3.2 ($m \geq n - 1$), Equation (12) becomes

$$2x_2 + 6x_3 z + \dots + n(n-1)x_n z^{n-2} = x_0 + x_1(q(z)) + \dots + x(q(z))^n \tag{19}$$

Where

$q(z) = (p_0 + bx_1) + \dots + (p_{n-1} + nbx_n)z^{n-1} + pnz^n + \dots + pmz^m$

Comparing coefficient of z^{nm} in (19) together with $p \neq 0$, we have $x_n = 0$. Then, repeating the above method.

We obtain $x(z) \equiv 0$ which is a contradiction.

Thus, (12) has a nontrivial polynomial solution $[x] = p_2$

with $p_0 \neq 0$ or $p(z) = p_0 + p_1 z$ with $p_i \neq 0$.

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