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An Analysis on Set-Theoretic Methods and Their Solution in Real Analysis

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Abstract – This article is a survey of the recent results that concern real functions and whose solutions or statements involve the use of set theory. The choice of the topics follows the author's personal interest in the subject, and there are probably some important results in this area that did not make it to this survey. Most of the results presented here are left without proofs.

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INTRODUCTION

The development of set theory created a new trend in mathematical research. On one hand it produced strong techniques such as transfinite recursion to solve long-standing open problems, and on the other hand the new theories enabled us to prove that it is impossible to answer certain questions; that is the usual axioms of set theory. Proving these so called consistent and independent statements is a very active and rapidly growing area of mathematics, specifically of analysis as well. My study presents a collection of my results of this type from the field of real analysis.

The study of real functions has played a fundamental role in the development of mathematics over the last three centuries. The seventeenth century discovery of calculus by Newton and Leibniz was largely due to increased understanding of the behavior of real functions. The birth of analysis is often traced to the early nineteenth century work of Cauchy, who gave precise definitions of concepts such as continuity and limits for real functions.

Convergence problems while approximating real functions by Fourier series gave rise to both the Riemann and Lebesgue integrals. Cantor developed his set theory in an effort to answer uniqueness questions about Fourier series. During this time, different techniques have been used as the theory behind them became available. For example, after Cauchy, various limiting operations such as pointwise and uniform convergence were studied, giving rise to various approximation techniques. At the turn of this century, measure theoretic techniques were exploited, leading to stochastic convergence ideas in the 1920's. Also, at about the same time topology was developed, and its applications to analysis gave rise to functional analysis. In recent years, a new research trend has appeared which indicates the emergence of a yet another branch of inquiry that could be called set theoretic real analysis.

Set theoretic real analysis is closely allied with descriptive set theory, but the objects studied in the two areas are different. The objects studied in descriptive set theory are various classes of (mostly nice) sets and their hierarchies, such as Borel sets or analytic sets. Set theoretic real analysis uses the tools of modern set theory to study real functions and is interested mainly in more pathological objects. Thus, the results concerning subsets of the real line (like the series of studies on \small" subsets of R, or deep studies of the duality between measure and category) are considered only remotely related to the subject.

Set theoretic real analysis already has a long history. Its roots can be traced back to the 1920's, where powerful new techniques based on the Axiom of Choice (AC) and the Continuum Hypothesis (CH) can be seen in many papers.

The new emergence of the field was sparked by the discovery of powerful new techniques in set theory and can be compared to the parallel development of set theoretic topology during the late 1950's and 1960's. In fact, it is a bit surprising that the development of set theoretic analysis is so much behind that of set theoretic topology, since at the beginning of the century the applicability of set theory in analysis was at least as intense as in topology.

This, however, can be probably attributed to the simple fact, that in the past half of a century there were many mathematicians that knew well both topology and set theory, and very few that knew well simultaneously analysis and set theory.

In analysis it is necessary to take limits; thus one is naturally led to the construction of the real numbers, a system of numbers containing the nationals and closed under limits. When one considers functions it is again natural to work with spaces that are closed under suitable limits. For example, consider the space of continuous functions *C*[0,1]. We might measure the size of a function here by

$$\|f\|_1 = \int_0^1 |f(x)|\, dx.$$

(There is no problem defining the integral, say using Riemann sums).

But we quickly see that there are Cauchy sequences of continuous functions whose limit, in this norm, are discontinuous. So we should extend C[0, 1] to a space that is closed under limits. It is not at first even evident that the limiting objects should be *functions.* And if we try to include *all* functions, we are faced with the difficult problem of integrating a general function.

The modern solution to this natural issue is to introduce the idea of *measurable functions,* i.e. a space of functions that is closed under limits and tame enough to integrate. The Riemann integral turns out to be inadequate for these purposes, so a new notion of integration must be invented. In fact we must first examine carefully the idea of the mass or *measure* of a subset $A \subset \mathbb{R}$, which can be thought of as the integral of its indicator function $\chi_A(x) = 1$ if $x \in A$ and = 0 if $x \notin A$.

3. *Fourier series.* More classical motivation for the Lebesgue integral come from Fourier series.

Suppose $f : [0, \pi] \to \mathbb{R}$ is a reasonable function. We define the Fourier coefficients of f by

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\, dx.$$

Here the factor of $2/\pi$ is chosen so that

$$\frac{2}{\pi} \int_0^\pi \sin(nx) \sin(mx)\, dx = \delta_{nm}.$$

We observe that if

$$f(x) = \sum_1^\infty b_n \sin(nx),$$

then at least formally a_n = b_n $a_n = b_n$ (this is true, for example, for a finite sum).

This representation of $f(x)$ as a superposition of sines is very useful for

applications. For example, $f(x)$ can be thought of as a sound wave, where a_n measures the strength of the frequency *n.*

Now what coefficients a_n can occur? The orthogonality relation implies that

$$\frac{2}{\pi} \int_0^\pi |f(x)|^2\, dx = \sum_{-\infty}^{\infty} |a_n|^2.$$

This makes it natural to ask if, conversely, for any a_n such that $\sum |a_n|^2 < \infty$, there exists a function f with these Fourier coefficients. The natural function to try is $f(x) = \sum a_n \sin(nx)$.

But why should this sum even exist? The functions sin*(nx)* are only bounded by one, and $\sum |a_n|^2 < \infty$ is much weaker than $\sum |a_n| < \infty$.

One of the original motivations for the theory of Lebesgue measure and integration was to refine the notion of function so that this sum really does exist. The resulting function $f(x)$ however need to be Riemann integrable! To get a reasonable theory that includes such Fourier series, Cantor, Dedekind, Fourier, Lebesgue, etc. were led inexorably to a re-examination of the foundations of real analysis and of mathematics itself. The theory that emerged will be the subject of this course.

SET THEORY

The foundations of real analysis are given by set theory, and the notion of cardinality in set theory, as well as the axiom of choice, occur frequently in analysis. Thus we begin with a rapid review of this theory. We then discuss the real numbers from both the axiomatic and constructive point of view. Finally we discuss open sets and Borel sets.

In some sense, real analysis is a pearl formed around the grain of sand provided by paradoxical sets. These paradoxical sets include sets that have no reasonable measure, which we will construct using the axiom of choice. The axioms of set theory. Here is a brief account of the axioms.

- Axiom I. (Extension) A set is determined by its elements. That is, if $x \in A \implies x \in B$ and vice-versa, then *A* = *B.*
- Axiom II. (Specification) If A is a set then $\{x \in A \;:\; P(x)\}$ is also a set.
- Axiom III. (Pairs) If *A* and *B* are sets then so is *{A.B}.* From this axiom and $\emptyset = 0$, we can now form {0,0} = {0}, which we call 1; and we can form {0,1}, which we call 2; but we cannot yet form {0,1,2}.

- Axiom IV. (Unions) If A is a set, then $\bigcup A = \{x : \exists B, B \in A \,\&\, x \in B\}$ is also a set. From this axiom and that of pairs we can form $\bigcup\{A, B\} = A \cup B.$ Thus we can define $x^+ = x + 1 = x \cup \{x\},$ and form, for example, 7 = {0,1,2,3,4,5,6}.

- Axiom V. (Powers) If A is a set, then $\mathcal{P}(A) = \{B : B \subset A\}$ is also a set.

- Axiom VI. (Infinity) There exists a set A such that $0 \in A$ and $x+1 \in A$ whenever $x \in A.$ The smallest such set is unique, and we call it $\mathbb{N} = \{0, 1, 2, 3, \ldots\}.$

- Axiom VII (The Axiom of Choice): For any set A there is a function $c : \mathcal{P}(A) - \{\emptyset\} \to A,$ such that $c(B) \in B$ for all $B \subset A.$

Cardinality. In set theory, the natural numbers N are defined inductively by $0 = \emptyset$ and $n = \{0, 1, \ldots, n-1\}.$ Thus n, as a set, consists of exactly *n* elements.

We write $|A| = |B|$ to mean there is a bijection between the sets *A* and *13:* in other words, these sets have the same *cardinality.* A set *A* is *finite* if $|A| = n$ for some $n \in \mathbb{N}$; it is *countable* if *A* is finite or $|A| = |\mathbb{N}|$; otherwise, it is *uncountable.*

A countable set is simply one whose elements can be written down in a (possibly finite) list, $(x_1, x_2, \ldots).$ When $|A| = |\mathbb{N}|$ we say *A* is *countably infinite.*

CARDINAL FUNCTIONS IN ANALYSIS

The important recent developments in set theoretical analysis concern the cardinal functions that are defined for different classes of real functions. These investigations seem to be analogous to those concerning of cardinal functions in topology from the 1970's and 1980's. (See [81, 77. 82, 152].) They are also related to the deep studies of cardinal invariants associated with different small subsets of the real line. (For a summary of the results concerning cardinals related to the measure and category see [65] or [8]. For a survey concerning cardinals associated with the thin sets derived from harmonic analysis see [18].)

The first group of funct ions is motivated by the notion of countable continuity and was introduced in 1991 by J. Cichori, M. Morayne, J. Pawlikowski, and S. Solecki in [22]. More precisely, they define the *decomposition* function $\mathrm{dec}(\mathcal{F}, \mathcal{G})$ for arbitrary families $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ and $\mathcal{G} \subset \bigcup\{\mathbb{R}^X : X \subset \mathbb{R}\},$ where Y^X stands for the set of all functions from *X to Y.*

$$\mathrm{dec}(\mathcal{F}, \mathcal{G}) = \min\{\kappa \le \mathfrak{c} : (\forall f \in \mathcal{F})(\exists \mathcal{X} \in \Pi_\kappa)(\forall X \in \mathcal{X})(f \restriction X \in \mathcal{G})\} \cup \{\mathfrak{c}^+\},$$

where Π_κ denotes the family of all coverings of $\mathbb{R}$ with at most κ many sets. In particular, if $\mathcal{C}$ stands for the family of all continuous functions (from subsets of $\mathbb{R}$ into $\mathbb{R}$) then $f : \mathbb{R} \to \mathbb{R}$ is countably continuous if and only if $\mathrm{dec}(\{f\}, \mathcal{C}) \le \omega.$

The motivation for this definition comes from a question of N. N. Luzin whether every Borel function is countable continuous. This question was answered negatively by P. S. Novikov and was subsequently generalized by Keldys (in 1934), and S. I. Adian and P. S. Novikov [l] (in 1958). The most general result in this direction was obtained in late 1980's by M. Laczkovich (see Cichori, Morayne) who proved, in particular, that $\mathrm{dec}(\mathcal{B}_\beta, \mathcal{B}_\alpha) > \omega$ for every $\alpha < \beta < \omega_1.$

One of the most interesting results from the paper [22] is the following theorem.

Theorem 4.1 (Cichori, Morayne, Pawlikowski, Solecki).

$$\mathrm{cov}(\mathcal{M}) \le \mathrm{dec}(\mathcal{B}_1, \mathcal{C}) \le d,$$

where $\mathrm{cov}(\mathcal{M})$ *is the smallest cardinality of a covering of* $\mathbb{R}$ *by meager sets, and d, the dominating number, is the smallest cardinality of a dominating family, i.e., such that for every* $f \in \omega^\omega$ *there exists* $g \in D$ *with* $f \le^* g_6^{D \subset \omega^\omega}$

It has been also shown by J. Steprans and S. Shelah that none of these inequalities can be replaced by the equation.

Theorem 4.2 (Steprans [147]). *It is consistent with ZFC that*

$$\mathrm{cov}(\mathcal{M}) < \mathrm{dec}(\mathcal{B}_1, \mathcal{C}).$$

Theorem 4.3 (Shelah, Steprans [134]). *It is consistent with ZFC that*

$$\mathrm{dec}(\mathcal{B}_1, \mathcal{C}) < d.$$

There are also some interesting results concerning the value of $\mathrm{dec}(\mathcal{C}, \mathcal{D}^1),$ where $\mathcal{D}^1$ is the class of all (partial) differentiable functions. It has been proved by Morayne (see Steprans [149, Thin 6.1]) that

Theorem 4.4 (Morayne [149, Thm 6.1]). $\mathrm{cov}(\mathcal{M}) \le \mathrm{dec}(\mathcal{C}, \mathcal{D}) \le \mathfrak{c}.$

Also, Steprans proved that

Theorem 4.5 (Steprans [149]). *It is consistent with ZFC that.*

$$\mathrm{dec}(\mathcal{C}, \mathcal{D}) < \mathfrak{c}.$$

However, the relation between numbers $\mathrm{dec}(\mathcal{C}, \mathcal{D})$, $\mathrm{dec}(\mathcal{B}_1, \mathcal{C})$ and $\mathrm{dec}(\mathcal{B}_\beta, \mathcal{B}_\alpha)$ for $0 < \alpha < \beta < \omega_1$ is unclear.

In the same direction, K. Ciesielski recently noticed that (obviously) $\mathrm{cf}(\mathfrak{c}) \leq \mathrm{dec}(\mathrm{SZ}, \mathcal{C}) \leq \mathfrak{c}$ and that it is the best that can be said in ZFC.

MEASURABLE FUNCTIONS

In this section we begin to study the interaction of measure theory with functions 011 the real line.

Theorem 4.1 *Given* $f : \mathbb{R} \to \mathbb{R}$, *the following conditions are equivalent.*

- $\{x : f(x) > a\}$ *is measurable for all* $a \in \mathbb{R}$.
- $f^{-1}(U)$ *is measurable for all open sets U.*
- $f^{-1}(B)$ *is measurable for all open Borel sets B.*

A function is *measurable* if any (and hence all) of these conditions hold. The first condition is the easiest to check.

Proof. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R})$ be the collection of sets $A \subset \mathbb{R}$ such that $f^{-1}(A)$ is measurable. Then *A* forms a a-algebra. Since the sets (a, ∞) generate the Borel sets as a σ-algebra, $(1) \implies (3)$. The implications $(3) \implies (2) \implies$ (1) are immediate.

First examples: continuous, monotone and indicator functions. Let $C(\mathbb{R})$ denote the space of all continuous functions on $\mathbb{R}$, and let $M(\mathbb{R})$ denote the set of all measurable functions 011 R. Clearly we have $C(\mathbb{R}) \subset M(\mathbb{R})$, since open sets are measurable.

In addition $M(\mathbb{R})$ contains monotone functions, since for these the preim age of an interval is another interval. The indicator functions χ_E of any measurable set is also easily shown to be measurable.

Algebraic structure. We now examine which operations we can form to make new measurable functions out of existing ones. It is well-known that $C(\mathbb{R})$ is an *algebra*, meaning if $f, g \in C(\mathbb{R})$ then so are $f + g$, fg and αf, $\alpha \in \mathbb{R}$.

Theorem 4.2 *The space* $M(\mathbb{R})$ *is an algebra, containing the continuous functions.*

Proof. If f is continuous and *U* is open, then $f^{-1}(U)$ is open, and hence measurable. Thus $C(\mathbb{R}) \subset M(\mathbb{R})$. It is clear that $M(\mathbb{R})$ is closed under scalar multiplication.

The tricky part is addition. Suppose $f, g \in M(\mathbb{R})$ and $f(x) + g(x) > a$. Then we can find a rational number p/q such that $f(x) > p/q$ and $p/q + g(x) > a$. (Just take p/q between $f(x)$ and $a - g(x)$.) and of course, this condition implies $f(x) + g(x) > a$. Thus we have:

$$\{x : f(x) + g(x) > a\} = \bigcup_{p/q \in \mathbb{Q}} \{x : f(x) > p/q\} \cap \{x : p/q + g(x) > a\}.$$

This expresses the set on the left as a countable union of measurable sets, so it is measurable.

As for products, we note that $(f + g)^2 - f^2 - g^2 = 2fg$, so it suffices to show that $M(\mathbb{R})$ is closed under $f \mapsto f^2$. This follows from the fact that

$$\{x : f(x)^2 > c^2\} = \{x : f(x) > c\} \cup \{x : f(x) < -c\}.$$

REFERENCES

1. Adian, S. I., Novikov, P. S. (1958). On one semicontinuous function, Uchen. Zap. MGPI W. I. Lenina 138, pp. 3-10. (In Russian.) Reprinted in: P. S. Novikov, Collected Papers. Set and Function Theory. Mathematical Logic and Algebra, \Nauka", Moscow, 1979. (In Russian.)
2. Balcerzak, M. (2002). Some remarks on sup-measurability, Real Anal. Exchange 17, pp. 597-607.
3. Bartoszynski, T. and Judah, H. (2005). Set Theory: on the structure of the real line, A. K. Peters.
4. Berarducci, A., Dikranjan, D. (2003). Uniformly approachable functions and UA spaces, Rend. Ist. Matematico Univ. di Trieste 25, pp. 23-56.
5. Brown, J. B. (2005). Restriction theorems in real analysis, Real Anal. Exchange 20(1), pp. 510-526.
6. H. Federer (2006). Geometric Measure Theory. Classics in Mathematics, Springer-Verlag.
7. Handbook of Set Theoretic Topology, edited by K. Kunen and J. E. Vaughan, North-Holland, 2004.
8. J. C. Oxtoby (2000). Measure and Category. Springer-Verlag.
9. P. Halmos (2004). Naive Set Theory. Springer-Verlag.

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