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On Parametric S-Metric Spaces and Fixed-Point Theorems for R-Weakly Commuting Mappings

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Abstract – In this paper, we prove some common fixed point theorems for variants of R-weakly commuting mappings, R-weakly commuting of type $(A_f),(A_g)$ and (P) in parametric S-metric spaces.

Keywords: parametric S-metric space; R-weakly commuting mappings; R-weakly commuting mappings of type $(A_f),(A_g)$ and (P)

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1. INTRODUCTION

In 1922 Stefan Banach proved a common fixed point theorem, which ensures the existence and uniqueness of a fixed point under appropriate conditions. This result of Banach is known as **Banach fixed point theorem or contraction mapping principle.**

These contractive conditions have been used in various fixed-point theorems for some generalized metric spaces.

Recently, the notion of an S-metric has been studied by some mathematicians. This notion was introduced by Sedghi et al. in 2012 [5] as follows,

Definition 1.1[5]Let *X* be a non-empty set and let *S*: $X\times X\times X \to [0,\infty)$ be a function. Then *S* is called an *S*-metric on X if,

(S1) $S(a,b,c)=0$ if an only if $a=b=c$,

(S2) $S(a,b,c)\le S(a,a,x)+S(b,b,x)+S(c,c,x)$, for each $a,b,c,x\in X$. The pair of (X,S) is called an *S*-metric space.

Definition 1.2 [3]Let *X* be a non-empty set and let $P: X\times X\times(0,\infty)\to[0,\infty)$ be a function.

P is called a parametric metric on *X* if,

(P1) $P(a,b,t)=0$ if and only if $a=b,$

(P2) $P(a,b,t)=P(b,a,t),$

(P3) $P(a,b,t)\le P(a,x,t)+P(x,b,t),$ for each $a,b,x\in X$ and all $t>0$. The pair of (X,P) is called a parametric metric space.

Definition 1.3 [3]Let (X,P) be a parametric metric space andlet $\{a_n\}$ be a sequence in *X* :

(1) $\{a_n\}$ converges to x if and only if there exist $n_0\in\mathbb{N}$ such that $P(a_n,x,t)<\varepsilon$, for all $n\ge n_0$ and all $t>0$; that is, $\lim_{n\to\infty}P(a_n,x,t)=0.$

(2) $\{a_n\}$ is called a Cauchy sequence if, for all $t>0$, $\lim_{n,m\to\infty}P(a_n,a_m,t)=0.$

(3) (X,P) is called complete if every Cauchy sequence is convergent.

Definition 1.4 Let *X* be a non- empty set and let $P_S: X\times X\times X\times(0,\infty)\to[0,\infty)$ be a function. P_S is called a parametric *S*-metric on *X* if,

(PS1) $P_S(a,b,c,t)=0$ if and only if $a=b=c,$

(PS2) $P_S(a,b,c,t)\le P_S(a,a,x,t)+P_S(b,b,x,t)+P_S(c,c,x,t),$ for each

$a,b,c \in X$ and all t>0. The pair (X,P_S) is called a parametric *S*-metric space.

Lemma 1.1 Let (X,P_S) be a parametric S-metric space.then we have

$P_S(a,a,b,t)=P_S(b,b,a,t),$ for each $a,b \in X$ and all t> 0.

Proof: using (PS2), we obtain,

$$P_S(a,a,b,t) \le 2P_S(a,a,b,t)+P_S(b,b,a,t)$$

$$= P_S(b,b,a,t),$$

$$P_S(b,b,a,t) \le 2P_S(b,b,b,t)+P_S(a,a,b,t)$$

$$= P_S(a,a,b,t).$$

From the above inequalities, we have

$$P_S(a,a,b,t)=P_S(b,b,a,t).$$

Definition 1.5 Let (X,P_S) be a parametric *S*-metric space and let $\{a_n\}$ be a sequence in *X*:

(1) $\{a_n\}$ converges to x if and only if there exists $n_0 \in \mathbb{N}$ such that $P_S(a_n,a_n,x,t)$ <ε,for all t>0; that is $\lim_{n\to\infty} P_S(a_n,a_n,x,t)=0.$

It is denoted by $\lim_{n\to\infty} a_n = x.$

(2) $\{a_n\}$ is called a Cauchy sequence if, for all t>0,

$$\lim_{n,m\to\infty} P_S(a_n,a_n,a_m,t)=0$$

(3) (X,P_S) is called complete if every Cauchy sequence is convergent.

Lemma 1.2 Let (X,P_S) be a parametric S-metric space. If $\{a_n\}$ converges to x, then x is unique.

Proof: Let $\lim_{n\to\infty} a_n = x$ and $\lim_{n\to\infty} a_n = y$ with $x \ne y$. Then there exists $n_1,n_2 \in \mathbb{N}$ such that

$P_S(a_n,a_n,x,t) < \frac{\varepsilon}{2}$,

$P_S(a_n,a_n,y,t) < \frac{\varepsilon}{2}$,

For each ε> 0, all t> 0, and $n > n_1,n_2$. if we take $n_0 = \max\{n_1,n_2\}$ then using (PS2) and lemma 1.1 , we get

$$P_S(x,x,y,t) \le 2P_S(x,x,a_n,t)+P_S(y,y,a_n,t)$$

$$< \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,$$

For each $n \ge n_0$. Therefore $P_S(x,x,y,t)=0$ and $x=y$.

2. FIXED POINT RESULT FOR *R*-WEAKLY COMMUTING MAPPINGS

In this section, we prove some fixed point results for *R*-weakly commuting mappings in parametric *S*-metric space.

Definition 2.1A pair (f,g) of self-mappings of a parametric *S*-metric space (X,P_S) is said to be *R*-weakly commuting at a point $a \in X$ if

$P_S(fga, fga, gfa, t) \le R\, P_S(fa, fa, ga, t)$ for some *R*> 0.

Definition 2.2 A pair (f,g) of self-mappings of a parametric *S*-metric space (X,P_S) is said to be point wise *R*-weakly commuting on *X* if given $a \in X$, there exists R >0 such that

$$P_S(fga, fga, gfa, t) \le R\, P_S(fa, fa, ga, t).$$

Theorem 2.1 Let (X,P_S) be a complete parametric *S*-metric space and let f and g be *R*-weakly commuting self-mapping of X satisfying the following conditions:

(i) $f(X) \subseteq g(X)$;

(ii) f or g is continuous;

(iii) $P_S(fa, fa, fb, t) \le q\ P_S(ga, ga, gb, t)$, for every $a, b \in X$ and $0 \le q < 1$.

Then f and g have a unique fixed point in X.

Proof: Let a_0 be an arbitrary point in X. By (i) one can choose a point $a_1 \in X$ such that $fa_0 = ga_1$. In general, choose a_{n+1} such that $b_n = fa_n = ga_{n+1}$.

Now we show that $\{b_n\}$ is a P_S - Cauchy sequence in X. For proving this we take $a = a_n, b = a_{n+1}$ in (iii), we have

$$P_S(fa_n, fa_n, fa_{n+1}, t) \le q\ P_S(ga_n, ga_n, ga_{n+1}, t) = q\ P_S(fa_{n-1}, fa_{n-1}, fa_n, t)$$

Continuing in the same way, we get

$$P_S(fa_n, fa_n, fa_{n+1}, t) \le q^n\ P_S(fa_0, fa_0, fa_1, t).$$

This implies,

$$P_S(b_n, b_n, b_{n+1}, t) \le q^n P_S(b_0, b_0, b_1, t).$$

Therefore, for all $n, m \in N$, $n < m$,

$$P_S(b_n, b_n, b_m, t) \le P_S(b_n, b_n, b_{n+1}, t) + P_S(b_{n+1}, b_{n+1}, b_{n+2}, t) + \ldots\ldots + P_S(b_{m-1}, b_{m-1}, b_m, t)$$

$$\le (q^n + q^{n+1} + \ldots\ldots + q^{m-1})\ P_S(b_0, b_0, b_1, t)$$

$$\le (q^n + q^{n+1} + \ldots\ldots\ldots\ldots\ldots)\ P_S(b_0, b_0, b_1, t)$$

$$= \frac{q^n}{(1-q)}\ P_S(b_0, b_0, b_1, t) \to 0 \text{ as } n \to \infty.$$

Thus $\{b_n\}$ is a P_S - Cauchy sequence in X. since (X, P_S) is complete S-metric space, therefore, there exists a point $c \in X$ such that $\lim_{n\to\infty} b_n = \lim_{n\to\infty} ga_n = \lim_{n\to\infty} fa_n = c$. Let us suppose that the mapping f is continuous. Therefore $\lim_{n\to\infty} fga_n = \lim_{n\to\infty} ffa_n = fc$.

Since f and g are R-weakly commuting,

$$P_S(fga_n, fga_n, gfa_n, t) \le R\ P_S(fa_n, fa_n, ga_n, t),$$

where R > 0.

On letting $n \to \infty$ we get $\lim_{n\to\infty} gfa_n = \lim_{n\to\infty} fga_n = fc$.

Now we prove that $c = fc$. Suppose $c \ne fc$ then $P_S(c, fc, fc, t) > 0$

On setting $a = a_n, b = fa_n$ in (iii), we have

$$P_S(fa_n, fa_n, ffa_n, t) \le q\ P_S(ga_n, ga_n, gfa_n, t)$$

Letting limit as $n \to \infty$ we get

$$P_S(c, c, fc, t) \le q\ P_S(c, c, fc, t) < P_S(c, c, fc, t).$$

Is a contradiction.

Therefore $c = fc$ since $f(X) \subseteq g(X)$ we can find $c_1 \in X$ such that $c = fc = gc_1$.

Now put $a = fa_n, b = c = c_1$ in (iii), we have

$$P_S(ffa_n, ffa_n, fc_1, t) \le q\ P_S(gfa_n, gfa_n, gc_1, t).$$

Taking limit as $n \to \infty$ we get

$$P_S(fc, fc, fc_1, t) \le q\ P_S(fc, fc, gc_1, t) = q\ P_S(fc, fc, fc, t) = 0,$$

Which implies that $fc = fc_1$ i.e., $c = fc = fc_1 = gc_1$. Also by using the definition og R-weakly commutativity,

$$P_S(fc, fc, gc, t) = P_S(fgc_1, fgc_1, gfc_1, t) \le R\ P_S(fc_1, fc_1, gc_1, t) = 0,$$

Implies that $fc = gc = c$. Thus c is a common fixed point of f and g.

Uniqueness: assume that $d(\ne c)$ be another common fixed point of f and g.

Then $P_S(c, c, d, t) > 0$ and

$P_S(c,c,d,t) = P_S(fc, fc, fd, t) \le q\ P_S(gc, gc, gd, t)$.

$= q\ P_S(c,c,d,t) < P_S(c,c,d,t)$.

Which is a contradiction. Therefore $c = d$. Hence uniqueness follows.

Example 2.1 Let $X = R$ and let the function $P_S : X \times X \times X \times (0,\infty) \to [0,\infty)$ be defined by

$P_S(a,b,c,t) = g(t)(|a-b| + |b-c| + |a-c|$ for each $a,b,c \in R$ and all t >0, where $g:(0,\infty) \to (0,\infty)$ is a continuous function. Then P_S is a parametric *S*-metric and the pair (R, P_S) is a parametric *S*-metric space. Define self-mappings *f* and *g* on *X* by $fx = x$ and $gx = 2x-1$, for all $x \in X$ .Here we note that,

(i) $f(X) \subseteq g(X)$;

(ii) *f* is continuous on *X* ;

(iii) $P_S(fa, fb, fc, t) \le q\ P_S(ga, gb, gc, t)$, holds for all $a,b,c \in X$, $\frac{1}{2} \le q$ <1.

Further, the mappings *f* and *g* are *R*-weakly commuting. Thus, all conditions of the theorem 2.1 are satisfied and *x* = 1 is the unique common fixed point of *f* and *g*.

3. FIXED POINT RESULT FOR VARIANTS OF *R*-WEAKLY COMMUTING MAPPINGS BY USING WEAK

Reciprocal continuity

In this section, we prove some fixed point results for *R*-weakly commuting mappings by using weak reciprocal continuity in parametric *S*-metric space.

Definition 3.1 Apair (f,g) of self-mappings of a parametric *S*-metric space (X, P_S) is said to be

(i) *R*-weakly commuting of type (A_g) if there exists some *R*>0 such that $P_S(ffa, ffa, gfa, t) \le R\ P_S(fa, fa, ga, t)$, for all $a \in X$.

(ii) *R*-weakly commuting of type (A_f) if there exists some *R*>0 such that $P_S(fga, fga, gga, t) \le R\ P_S(fa, fa, ga, t)$, for all $a \in X$.

Definition 3.2 A pair (f,g) of self-mappings of a parametric *S*-metric space (X, P_S) is said to be *R*-weakly commuting of type (*P*) if there exists some *R*>0 such that

$P_S(ffa, ffa, gga, t) \le R\ P_S(fa, fa, ga, t)$, for all $a \in X$.

Theorem 3.1let f and g be weakly reciprocally continuous self-mappings of a complete parametric S-metric space (X, P_S) satisfying the following conditions:

(i) $f(X) \subseteq g(X)$;

(ii) $P_S(fa, fa, fb, t) \le q\ P_S(ga, ga, gb, t)$, for every $a,b \in X$ and $0 \le q$ <1.

If f and g are either compatible or *R*-weakly commuting of type (A_g) or *R*-weakly commuting type (A_f) or *R*-weakly commuting of type (*P*), then f and g have a unique common fixed point.

Proof. Let a_0 be an arbitrary point in *X*. As $f(X) \subseteq g(X)$, therefore there exists a sequence of points $\{a_n\}$ such that $fa_n = ga_{n+1}$.

Define a sequence $\{b_n\}$ in *X* by $b_n = fa_n = ga_{n+1}$. (3.1)

Sequence $\{b_n\}$ is P_S - Cauchy sequence in *X* . (the proof follows the same lines as in Theorem 2.1). Now since (X, P_S) is complete parametric S-metric space, therefore there exists a point $c \in X$ such that $\lim_{n\to\infty} b_n = c$.

Hence $\lim_{n\to\infty} b_n = \lim_{n\to\infty} fa_n = \lim_{n\to\infty} ga_n = c$. Suppose that f and g are compatible mappings. Weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fga_n = fc$ or $\lim_{n\to\infty} gfa_n = gc$. Let

$\lim_{n\to\infty} gfa_n = gc$ then the compatibility of f and g gives:

$\lim_{n\to\infty} P_S(fga_n, fga_n, gfa_n, t) = 0$ i.e., $P_S(\lim_{n\to\infty} fga_n, fga_n, gc, t) = 0$. Hence, $\lim_{n\to\infty} fga_n = gc$.

By (2.1) we have $\lim_{n\to\infty} fga_{n+1} = \lim_{n\to\infty} ffa_n = gc$.

Therefore by the use of (ii) we get

$$P_S(fc, fc, ffa_n, t) \le q\ P_S(gc, gc, gfa_n, t).$$

Letting $n \to \infty$ on both sides we have

$$P_S(fc, fc, gc, t) \le q\ P_S(gc, gc, gc, t) = 0$$

This gives $fc = gc$. Again compatibility of f and g implies commutativity at a coincidence point. Hence $gfc = fgc = ffc = ggc$. Now, we claim that $fc = ffc$. Suppose $fc \ne ffc$, then by using (ii) we obtain,

$$P_S(fc, fc, ffc, t) \le q\ P_S(gc, gc, gfc, t)$$

$$P_S(fc, fc, ffc, t) \le q\ P_S(fc, fc, ffc, t),$$

Which is a contradiction Since $q \in [0,1)$. Hence, $fc = ffc = gfc$ and fc is a common fixed point of f and g.

Next suppose that $\lim_{n\to\infty} fga_n = fc$. Then $f(X) \subseteq g(X)$ implies $fc = gu$ for some $u \in X$ and therefore $\lim_{n\to\infty} fga_n = gu$.

Compatibility of f and g implies, $\lim_{n\to\infty} gfa_n = gu$.

By virtue of (3.1), we have $\lim_{n\to\infty} fga_{n+1} = \lim_{n\to\infty} ffa_n = gu$.

Using (ii), we get

$$P_S(fu, fu, ffa_n, t) \le q\ P_S(gu, gu, gfa_n, t).$$

Letting $n \to \infty$ on both sides, we have

$P_S(fu, fu, gu, t) \le q\ P_S(gu, gu, gu, t) = 0$,

This gives $fu = gu$. Compatibility of f and g yields $fgu = ggu = ffu = gfu$. Finally we claim that $fu = ffu$. suppose that $fu \ne ffu$, then by using (ii) we obtain,

$$P_S(fu, fu, ffu, t) \le q\ P_S(gu, gu, gfu, t)$$

$$P_S(fu, fu, ffu, t) \le q\ P_S(fu, fu, ffu, t)$$

which again gives a contradiction, since $q \in [0,1)$. Hence $fu = ffu = gfu$. Therefore fu is a common fixed point of f and g.

Now suppose that f and g are *R*-weakly commuting type of (A_g). Now, weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fga_n = fc$ or $\lim_{n\to\infty} gfa_n = gc.$ Let us first assume that $\lim_{n\to\infty} gfa_n = gc.$ Then *R*-weakly commutativity of type (A_g) of f and g yields

$P_S(ffa_n, ffa_n, gfa_n, t) \le R\ P_S(fa_n, fa_n, ga_n, t)$ where *R*>0.

Letting $n \to \infty$ on both sides, we have

$$P_S(\lim_{n\to\infty} ffa_n, ffa_n, gc, t) \le R\ P_S(c, c, c, t) = 0.$$

This gives $\lim_{n\to\infty} ffa_n = gc.$ Also using (ii) we get

$$P_S(fc, fc, ffa_n, t) \le q\ P_S(gc, gc, gfa_n, t).$$

Letting $n \to \infty$ on both sides we have

$$P_S(fc, fc, gc, t) \le q\ P_S(gc, gc, gc, t) = 0.$$

Hence we get $fc = gc$. Again by using *R*-weakly commutativity of type (A_g),

$$P_S(ffc, ffc, gfc, t) \le R\ P_S(fc, fc, gc, t) = 0$$

This yields $ffc = fgc$. Therefore, $ffc = fgc = gfc = ggc$. We claim that $fc = ffc$. Suppose $fc \neq ffc$

Using (ii) we get

$$P_S(fc, fc, ffc, t) \leq q\ P_S(gc, gc, gfc, t)$$

$$P_S(fc, fc, ffc, t) \leq q\ P_S(fc, fc, ffc, t)$$

A contradiction. Hence $fc = ffc = gfc$ and fc is common fixed point of f and g.

Similarly, we an prove if $\lim_{n\to\infty} fga_n = fc.$

On the other hand if f and g are *R*-weakly commuting mappings of type (A_f), then by following the similar steps as presented above, it can easily be proved that fc is a common fixed point of f and g.

Finally now, suppose that f and g are *R*-weakly commuting of type (*P*). Weak reciprocal continuity of f and g implies that $\lim_{n\to\infty} fga_n = fc$ or $\lim_{n\to\infty} gfa_n = gc.$ Let us assume that $\lim_{n\to\infty} gfa_n = gc.$ since pair (f, g) *R*-weakly commuting of type (*P*), we have

$$P_S(ffa_n, ffa_n, gga_n, t) \leq R\ P_S(fa_n, fa_n, ga_n, t)$$

where R >0.

Letting $n \to \infty$ on both sides we get,

$$P_S(\lim_{n\to\infty} ffa_n, \lim_{n\to\infty} ffa_n, \lim_{n\to\infty} gga_n, t) \leq R\ Ps(c, c, c, t) = 0.$$

This gives $P_S(\lim_{n\to\infty} ffa_n, \lim_{n\to\infty} ffa_n, \lim_{n\to\infty} gga_n, t) = 0$.

Using (i) and (3.1) we have

$gfa_{n-1} = gga_n \to gc$ as $n \to \infty$ this gives, $ffa_n \to gc$ $n \to \infty$. Also, by using (ii) we have

$$P_S(fc, fc, ffa_n, t) \leq q\ P_S(gc, gc, gfa_n, t)$$

Letting $n \to \infty$ on both sides we get

$$P_S(fc, fc, gc, t) \leq q\ P_S(gc, gc, gc, t) = 0.$$

This implies that $fc = gc$. Again by using *R*-weakly commutativity of type (*P*),

$P_S(ffc, ffc, ggc, t) \leq R\ P_S(fc, fc, gc, t) = 0$ where *R*>0 .

This yields $fc = ggc$. Therefore $ffc = fgc = gfc = ggc.$

Lastly, we claim that $fc = ffc.$ Suppose that $fc \neq ffc$.

Using (ii) we have

$$P_S(fc, fc, ffc, t) \leq q\ P_S(gc, gc, gfc, t)$$

$$P_S(fc, fc, ffc, t) \leq q\ P_S(fc, fc, ffc, t),$$

Which is a contradiction. Therefore $fc = ffc.$

Hence $fc = ffc = gfc$ and fc is a common fixed point of f and g.

This results holds good even if $\lim_{n\to\infty} fga_n = fc$ is considered instead of $\lim_{n\to\infty} gfa_n = gc.$

Uniqueness of the common fixed point in each of the three types of mappings can easily be obtained by using (ii).

The following example shows the validity of theorem 3.1.

Example 3.1 Let (X, P_S) be S-metric space, where X = [2,20] and for all $a, b, c \in X$ $P_S(|a-b| + |b-c| + |a-c|)$. Define *f*, *g* : $X \to X$ by $fa = \begin{cases} 2, & a = 2 \text{ or } a > 5 \\ 6, & 2 < a \leq 5 \end{cases}$ and

$ga = \begin{cases} 2, & a = 2 \\ 12, & 2 < a \leq 5 \\ \frac{x+1}{3}, & a > 5 \end{cases}$ it can be easily verified that

(i) $f(X) \subseteq g(X)$;

(ii) *f* and *g* satisfies condition (ii) of theorem 3.1;

(iii) *f* and *g* are *R*-weakly commuting of type $\left(A_g\right)$;

(iv) *f* and *g* are weakly reciprocally continuous for sequences $\{a_n\} = \{2\}$ or $\{a_n\} = \left\{5 + \frac{1}{n}\right\}$ for each in X.

Thus, *f* and *g* satisfy all the conditions of theorem 3.1 and have a unique common fixed point at $x = 2$.

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