Journal of Advances and Scholarly Researches in Allied Education
Vol. XIV, Issue No. 2, January-2018, ISSN 2230-7540

An Analysis on Fix Points Common and Natural Families of Some Homogeneous Polynomials

Kate Sunil Krishnaji[1]* Dr. Sanjay Kumar Gupta[2]

[1] PhD Student, MUIT, Lucknow

[2] Assistant Professor and PhD Guide, MUIT, Lucknow

Abstract – The present study research the fixed points of meromorphic functions, and their higher order contrasts and moves, and sum up the instance of fixed points into the more broad case for first order distinction and move. Solidly, some appraisals on the order and the types of assembly of uncommon points of meromorphic functions and their disparities and movements are gotten.

Keywords: Fix Points, Natural Families, Homogeneous Polynomials

- X -

INTRODUCTION

The study of Fix points plays an important role in the criteria of Normality of meromorphic functions. The connection between Fix-points and Normality criteria was given by the following theorem of Yong Lo.

Theorem: Let F be a meromorphic function family in area D, and let k be a positive integer. If for some f(z) of F function, Both f and f and $f^{(k)}(z)$ Don't have fix points in D, Then there is regular F.

C.M. Hombale extended this result to certain homogeneous differential polynomials and proved the following theorem

Theorem : Let F be a family of meromorphic functions in a region D, k be positive integer if

i) For every $f \in F$, f has only multiple poles and at every double pole z0, the Laurent expansion of f(z) has the form

$$f(z) = \frac{a}{(z-z_0)^2} + 0(1), \quad a \neq 0$$

ii) For every $f \in F$, $f^2(z)$ and $\left(\frac{f^2(z)}{2}\right)^{(k)}$ (the derivative of order k) Don't have fix points in D, Then there is regular F.

Here, In that segment, We don't just expand the theorem B above, but, also remove the condition (i), we adopt a different technique, and here we prove,

Theorem: Let F be a non-zero meromorphic family of functions in a region D, $n \geq 2$, K are positive results, if any $f \in F, (f^n(z))^{(k)}$ (the k^{th} derivative of order f(z)) Don't have repair points in D, then F's natural.

Let

$$\phi(z) = \left(f^n(z)\right)^{(k)} \qquad \dots(1)$$

Lemma: Let f(z) have meromorphic in $|z| < R \ (0 < R \leq \infty)$. If $f(0) \neq 0, \phi(0) \neq d, d\phi'(0) - \phi(0) \neq 0$ after that, They've got

$$nT(r,f) \leq \overline{N}(r,f) + n\,N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-N\left(r,\frac{1}{(z+d)\phi'-\phi}\right) + S(r,f)$$

Proof: Consider the identity

$$\frac{1}{f^n} = \frac{\phi}{f^n(z+d)} - \frac{(z+d)\phi'-\phi}{(z+d)f^n} \cdot \frac{\phi-(z+d)}{(z+d)\phi'-\phi} \qquad \dots(2)$$

This leads to

$$m\left(r,\frac{1}{f^n}\right)\le m\left(r,\frac{\phi}{f^n}\right)+m\left(r,\frac{1}{(z+d)}\right)+m\left(r,\frac{\phi'}{f^n}\right)$$

$$+m\left(r,\frac{1}{(z+d)}\right)+m\left(r,\frac{\phi}{f^n}\right)$$

$$+m\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi}\right)+C$$

Now since

$$\phi=(f^n)^{(k)}$$

Hence

$$m\left(r,\frac{\phi}{f^n}\right)=S(r,f^n)=S(r,f).$$

Similarly

$$m\left(r,\frac{\phi'}{f^n}\right)=S(r,f)$$

And

$$m\left(r,\frac{1}{z+d}\right)=O(\log r)$$

Hence we obtain

$$m\left(r,\frac{1}{f^n}\right)\le m\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi}\right)+S(r,f)$$

Adding $N\left(r,\frac{1}{f^n}\right)$ on both sides, and by using first fundamental theorem of Nevanlinna, we note that

$$nT(r,f)\le m\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi}\right)+nN\left(r,\frac{1}{f}\right)+\log\left|f^n(0)\right|+S(r,f)$$

Thus, we obtain

$$nT(r,f)\le m\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi}\right)+nN\left(r,\frac{1}{f}\right)+S(r,f) \qquad \ldots(3)$$

Now

$$m\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi}\right)=T\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi}\right)-N\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi}\right)$$

$$\le m\left(r,\frac{(z+d)\phi'-\phi}{\phi-(z+d)}\right)+N\left(r,\frac{(z+d)\phi'-\phi}{\phi-(z+d)}\right)$$

$$-N\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi)}\right)-\log\left|\frac{d\phi'(0)-\phi(0)}{\phi(0)-d}\right|$$

$$\le m\left(r,\frac{(z+d)\phi'-(z+d)-(\phi-(z+d))}{\phi-(z+d)}\right)$$

$$+N\left(r,\frac{(z+d)\phi'-\phi}{\phi-(z+d)}\right)-N\left(r,\frac{\phi-(z+d)}{(z+d)\phi'-\phi}\right)$$

$$+\log\left|\frac{\phi(0)-d}{d\phi'(0)-\phi(0)}\right|+C$$

$$\le N(r,(z+d)\phi'-\phi)+N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-N(r,\phi-(z+d))-N\left(r,\frac{1}{(z+d)\phi'-\phi}\right)+S(r,f)$$

$$\le\overline{N}(r,f)+N\left(r,\frac{1}{\phi-(z+d)}\right)-N\left(r,\frac{1}{(z+d)\phi'-\phi}\right)+S(r,f) \qquad \ldots(4)$$

Thus, by combining, (4) and (3), we get

$$nT(r,f)\le\overline{N}(r,f)+nN\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-N\left(r,\frac{1}{(z+d)\phi'-\phi}\right)+S(r,f),$$

Where

$$S(r,f)=2m\left(r,\frac{\phi}{f^n}\right)+m\left(r,\frac{\phi'}{f^n}\right)+m\left(r,\frac{\phi'-1}{\phi-(z+d)}\right)+\log\left|\frac{\phi(0)-d}{d\phi'(0)-\phi(0)}\right|+C$$

Lemma: Suppose, f(z) is as given in lemma, and

$$f(0)\ne 0,\infty \quad \phi(0)\ne d,\quad d\phi'(0)-\phi(0)\ne 0$$

Then

$$T(r,f)<\frac{n}{(n-1)}N\left(r,\frac{1}{f}\right)+\frac{1}{(n-1)}N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-\frac{1}{(n-1)}N\left(r,\frac{1}{\phi'(z+d)-\phi}\right)+S(r,f)$$

Proof: Since $n \geq 2$ and by lemma, we have

$$T(r,f) \leq \frac{1}{n}\overline{N}(r,f) + N\left(r,\frac{1}{f}\right) + \frac{1}{n}N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-\frac{1}{n}N\left(r,\frac{1}{(z+d)\phi'-\phi}\right) + S(r,f) \qquad \ldots(5)$$

But, since

$\overline{N}(r,f) \leq T(r,f)$ We have

$$\left(1-\frac{1}{n}\right)\overline{N}(r,f) \leq N\left(r,\frac{1}{f}\right) + \frac{1}{n}N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-\frac{1}{n}N\left(r,\frac{1}{(z+d)\phi'-\phi}\right) + S(r,f)$$

Thus

$$\frac{1}{n}\overline{N}(r,f) \leq \frac{1}{(n-1)}N\left(r,\frac{1}{f}\right) + \frac{1}{n(n-1)}N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-\frac{1}{n(n-1)}N\left(r,\frac{1}{(z+d)\phi'-\phi}\right) + S(r,f) \qquad \ldots(6)$$

Substituting this inequality in (5) we get

$$T(r,f) \leq \left(1+\frac{1}{n-1}\right)N\left(r,\frac{1}{f}\right) + \frac{1}{n}\left(1+\frac{1}{n-1}\right)N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-\frac{1}{n}\left(1+\frac{1}{n-1}\right)N\left(r,\frac{1}{(z+d)\phi'-\phi}\right) + S(r,f)$$

Thus

$$T(r,f) \leq \frac{n}{(n-1)}N\left(r,\frac{1}{f}\right) + \frac{1}{(n-1)}N\left(r,\frac{1}{\phi-(z+d)}\right)$$

$$-\frac{1}{(n-1)}N\left(r,\frac{1}{(z+d)\phi'-\phi}\right) + S(r,f),$$

Lemma: Suppose f(z) satisfies the assumptions of Lemma, with $R < \infty$ and in addition, $f(z) \neq 0,\ \phi(z) \neq (z+d),\ d \neq 0$ in $|z| < R$, then, we have

$$\log M\left(r,\frac{1}{f}\right) < C\frac{R}{R-r}\left(1+B+\log\frac{R}{R-r}\right)$$

for $0 < r < R$ where C is a positive numerical constant and

$$B = \log^+\log^+\frac{1}{|f(0)|} + \log^+|f(0)| + \log^+|\phi(0)-d| + \log\frac{1}{|d\phi'(0)-\phi(0)|} + C$$

Proof: By hypothesis, we note that

NORMALITY CRITERIA FOR A FAMILY OF MEROMORPHIC FUNCTION HAVING FINITELY MANY SIMPLE POLES

We have proved a result on The singularity of meromorphic functions. But we do prove in this part a result concerning normality criterion for a family of meromorphic functions. We have

Theorem: Let F be a family of meromorphic functions having only multiple poles in D, such that for each $f \in F.\ f^n f'(f-1) \neq 0$ in D for any fixed integer $n > 0$. In D, then, F is normal.
We need the following lemmas for proof

Lemma: (Heiong estimate) Suppose that f(z) is meromorphic in $|z| \leq R\ 0 < r < \rho < R$ and that. $f(0) \neq 0, \infty$ Then,

$$m\left(r,\frac{f^{(k)}}{f-a}\right) \leq C\left\{\log^+ T(\rho,f) + \log^+\rho + \log^+\frac{1}{\rho-r} + \log^+\frac{1}{r}\right.$$

$$\left.+\log^+\log^+\frac{1}{|f(0)-a|}+1\right\}$$

For $0 < r < \rho \leq R$

Holds for every pair of $r, \rho, [R_1 < r < \rho < R_2]$ then we have

Lemma: Suppose $(f_j), j \in N$ is a sequence of meromorphic functions on $|z| < \delta$ and $f_j(0) \neq \infty$ for all $j \in N$. if $T(r_0, f_j) < C$ for $r_0 \in (0,\delta), C \in R$; and $j \in N$. then there exists a neighborhood of 0 on which some subsequence of $(f_j)_{j\in N}$ tends to an analytic function.

GENERALISED CRITERION FOR NORMAL FAMILIES

Let f (z) is a meromorphic function in a domain D. We assume familiarity with usual notations of Nevanlinna theory. Throughout this we use $C, C_1, C_2 \ldots$ to denote the fixed constants depending at most on a, b and R. As is standard, we define, $B(z_0, R) = \{z \in D; |z - z_0| < R\}$.

We also let $M(z) = (f)^{l_0}(f')^{l_1}\ldots\left(f^{(k)}\right)^{l_k}$ where $l_0, l_1 \ldots l_k$ are non-negative integers, we call $\gamma = l_0 + l_1 + \ldots + l_k$ as the degree of the monomial M(z) and $\Gamma = l_0 + 2l_1 + \ldots + (k+1)l_k$ as it's weight.

In the theory of normal families, A major problem is finding new standards of normality. Nevanlinna theory plays a significant role in that regard. The following theorem was proved by Langley. J.K using the Nevanlinna theory.

Let F be a meromorphic family of functions in domain D and for fixed functions $n \geq 5$, $a \neq 0$, $f'(z) - af^n(z) \neq b$, $\forall f \in F$, and for some $b \in C$, then F is normal in D.

We prove a significant generalization of Langley's result. For the proof, we use the methods of both Langley.J.K, Xu.Y and X.Hua, This approach greatly simplifies the proofs. They show the following principal theorem.

Theorem: Let F become a meromorphic family of functions within a domain. Suppose a, b are complex and finite numbers with $a \neq 0$ and for fixed integer $n \geq \Gamma + 3$, for each $f \in F$, $M(z) - aP[f(z)] \neq b$, where $P[f(z)] = a_1 f^n(z) + a_2 f^{n-1}(z) + \ldots + a_n f(z)$ with $|a_1| \geq 1$ F in D then is natural.

Lemmas

Lemma: Let F be a meromorphic function family within a Domain D, such that for each $f \in F$, $M(z) - aP[f(z)] = b$ has no solutions there (where a, b are finite, $a \neq 0$ and $n \geq \Gamma + 3$) Suppose further that $f(0) \neq 0, \infty$, $K(0) \neq \infty$ and $K'(0) \neq 0, \infty$; where

$$K(z) = \frac{M(z) - b}{aP[f(z)]}$$

Then for $0 < r < s < R$.

$$T(r,f) < D\left(\log^+ s + \log^+ \frac{1}{s-r} + \log^+ \frac{1}{r}\right) + E$$

Proof: - Since

$$K(z) = \frac{M(z) - b}{aP[f(z)]}, \qquad aP[f(z)] = \frac{M(z) - b}{K(z)}$$

Hence,

$$nm(r,f) \sim m(r, P[f(z)]) \leq m(r, aP[f(z)]) + \log^+ \frac{1}{|a|} + C$$
$$\leq m(r, M) + m\left(r, \frac{1}{K}\right) + \log^+ \left|\frac{b}{a}\right| + C$$

Therefore,

$$nm(r,f) \leq m(r,(f)^{l_0}) + m(r,(f')^{l_1}) + \ldots + m(r,(f^{(k)})^{l_k}) + m\left(r, \frac{1}{K}\right) + C$$
$$\leq l_1 m\left(r, \frac{f'}{f}\right) + l_2 m\left(r, \frac{f''}{f}\right) + \ldots + l_k m\left(r, \frac{f^{(k)}}{f}\right) + (l_0 + l_1 + \ldots + l_k) m(r,f) + m\left(r, \frac{1}{K}\right) + C$$

Noting that $\gamma = (l_0 + l_1 + \ldots + l_k)$,

We have,

$$(n - \gamma)\, m(r,f) \leq m\left(r, \frac{1}{K}\right) + l_1 m\left(r, \frac{f'}{f}\right) + \ldots + l_k m\left(r, \frac{f^{(k)}}{f}\right) + C \qquad \ldots(1)$$

Using the standard estimate (5) for

$$m\left(r, \frac{f^{(l)}}{f}\right), \; l = 1, 2, \ldots\ldots k.$$

we have

$$(n - \gamma)\, m(r,f) \leq m\left(r, \frac{1}{K}\right) + A \log^+ T(t,f) + B\left(\log^+ t + \log^+ \frac{1}{t-r} + \log^+ \frac{1}{r}\right) + C_1$$
$$+ \log^+ \log^+ \frac{1}{|f(0)|} + C \quad \text{for } 0 < r < t < R. \qquad \ldots(2)$$

Also,

$$\bar{N}\left(r, \frac{1}{K}\right) = \bar{N}(r,f) + \bar{N}\left(r, \frac{1}{M-b}\right).$$

If we denote by $N_0(r)$ the counting function of the common zeros of both K and M(z) - b, we rewrite the above equation as,

$$\bar{N}\left(r, \frac{1}{K}\right) = \bar{N}(r,f) + \bar{N}_0(r) \qquad \ldots(3)$$

Where in $N_0(r)$, we count the distinct common zeros of K and M(z) - b.

Obviously, A zero of K is either a pole of f or a zero of M(z)-b and a pole of f of order p must be a zero of K of order p $np - [l_0 p + (p+1)l_1 + \ldots + (p+k)l_k] \geq n - \Gamma$. Hence,

$$N\left(r, \frac{1}{K}\right) \geq (n - \Gamma) N(r,f) + N_0(r)$$

So,

$$(n - \Gamma) N(r,f) \leq N\left(r, \frac{1}{K}\right) - N_0(r) \qquad \ldots(4)$$

Combining (2), (4) and using Nevanlinna's second fundamental theorem

$$\begin{aligned}(n-\Gamma)\,T(r,f) &\le T\left(r,\frac{1}{K}\right)-N_0(r)+A\log^+T(t,f)\\ &+B\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)+C_1\log^+\log^+\frac{1}{|f(0)|}+C\\ &\le \bar{N}\left(r,\frac{1}{K}\right)+\bar{N}(r,K)+\bar{N}\left(r,\frac{1}{K-1}\right)\\ &+2m\left(r,\frac{K'}{K}\right)+m\left(r,\frac{K'}{K-1}\right)+\log\left|\frac{K(0)[K(0)-1]}{K'(0)}\right|\\ &-N_0(r)+A\log^+T(t,f)+B\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)\\ &+C_1\log^+\log^+\frac{1}{|f(0)|}+C\end{aligned}$$

As $M(z)-aP[f(z)]\ne b,\ \forall z\in D,\ \bar{N}\left(r,\frac{1}{K-1}\right)=0$. Also it is easy to see that, $\bar{N}(r,K)\le\bar{N}\left(r,\frac{1}{f}\right)$ From the above discussion and by equation (3),

We have,

$$\begin{aligned}(n-\Gamma)\,T(r,f) &\le \bar{N}(r,f)+\bar{N}_0(r)+N\left(r,\frac{1}{f}\right)+2m\left(r,\frac{K'}{K}\right)+m\left(r,\frac{K'}{K-1}\right)-N_0(r)\\ &+A\log^+T(t,f)+B\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)\\ &+C_1\log^+\log^+\frac{1}{|f(0)|}+\log\left|\frac{K(0)[K(0)-1]}{K'(0)}\right|+C\end{aligned}$$

This gives,

$$\begin{aligned}(n-\Gamma)\,T(r,f) &\le N(r,f)+N\left(r,\frac{1}{f}\right)+2m\left(r,\frac{K'}{K}\right)+m\left(r,\frac{K'}{K-1}\right)\\ &+A\log^+T(t,f)+B\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)\\ &+C_1\log^+\log^+\frac{1}{|f(0)|}+\log\left|\frac{K(0)[K(0)-1]}{K'(0)}\right|+C\end{aligned}$$

Now using the standard estimate [5] for $m\left(r,\frac{K'}{K}\right)$ and $m\left(r,\frac{K'}{K-1}\right)$,

We can write

$$\begin{aligned}(n-\Gamma-2)\,T(r,f) &\le A\log^+T(t,f)+B\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)\\ &+C_1\log^+\log^+\frac{1}{|f(0)|}+A\log^+T(t,K)\\ &+B\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)\\ &+C_2\log^+\log^+\frac{1}{|K(0)|}+C_3\log^+\log^+\frac{1}{|f(0)-1|}\\ &+\log\left|\frac{K(0)[K(0)-1]}{K'(0)}\right|+C\quad\text{for } 0<r<t<R. \qquad \ldots(5)\end{aligned}$$

Now since, $K(z)=\frac{M(z)-b}{aP[f(z)]}$, we have, (for $0<r<t<R$)

$$\begin{aligned}T(r,K) &\le T(r,M)+T(r,P[f])+\log\frac{1}{|P[f(0)]|}+C\\ &< (n+\Gamma)T(r,f)+\log\frac{1}{|P[f(0)]|}+A\log^+T(t,f)\\ &+B\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)+C_1\log^+\log^+\frac{1}{|f(0)|}+C\end{aligned}$$

Therefore,

$$\begin{aligned}A\log^+T(t,K) &< A_1\log^+T(t,f)+B\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)\\ &+\log^+\log^+\frac{1}{|P[f(0)]|}+C_1\log^+\log^+\frac{1}{|f(0)|}+C\end{aligned}$$

Substituting this in equation (5), we get,

$$(n-\Gamma-2)\,T(r,f)\le A_2\log^+T(t,f)+B_2\left(\log^+t+\log^+\frac{1}{t-r}+\log^+\frac{1}{r}\right)+E_0$$

Where

$$\begin{aligned}E_0 &= C_1\log^+\log^+\frac{1}{|f(0)|}+\log^+\log^+\frac{1}{|P[f(0)]|}+\log^+|K(0)|\\ &+\log^+|K(0)-1|+\log\frac{1}{|K'(0)|}\end{aligned}$$

Now applying Bureau's lemma in the interval (0,S) we obtain,

$$(n-\Gamma-2)\,T(r,f)\le D_1\left(\log^+S+\log^+\frac{1}{S-r}+\log^+\frac{1}{r}\right)+E_1$$

$$\text{for } 0<r<t<S<R.$$

We need lemma due to Ku

Lemma: Suppose that f (z) is meromorphic in $|z|<R$ with $f(0)\ne\infty$. Suppose that $T(r,f)<m$ for $0<r<R$. Then f is regular in $|z|<e^{-m}R$ and $|f(z)|<e^{5m}$ in $|z|\le\frac{1}{2}e^{-m}R$.

Proof of Theorem: Given a point $z_0\in D$, we take a positive R such that $B(z_0,R)\subset D$. Without loss of generality we may assume that $z_0=0$. We set $L=D\left(\log^+R+2\log^+\frac{2}{R}\right)+E$ where D and E are constants of by Lemma. Now by Lemma, we conclude that,|

$$T(r,f)<D\left(\log^+R+\log^+\frac{1}{R-r}+\log^+\frac{1}{r}\right)+E.$$

We take $r=\frac{R}{2}$ and hence,

$$T\left(\frac{R}{2},f\right)<D\left(\log^{+}R+2\log^{+}\frac{2}{R}\right)+E=L$$

And so by lemma, f (z) is regular in $|z|<e^{-L}R$ and $|f(z)|<e^{5L}$ in $|z|<\frac{1}{2}e^{-L}R$.

We settle for a small positive δ so that $\delta R<1$. And so $|f(z)|<(5L)$ in $B\left(0,\frac{\delta R}{2}\right)$. Hence by Montel's theorem, we conclude that F is normal in D.

Further we obtain the following result of Langley as a special case of theorem.

Corollary: Let F be a family of meromorphic functions in a domain D, for fixed $n\geq 5$, a, b are finite complex numbers with $a\neq 0$ and $f'(z)-af^{n}(z)=b$ has no solution in D, then F is normal in D.

Proof of Corollary: Here $M(z)=f'(z)$, $P[f(z)]=f^{n}(z)$, we note that $\gamma=1, \Gamma=2$. Therefore, taking $n\geq\Gamma+3=5$; we arrive at the result of Langley.

NORMALITY CRITERIA CONCERNING DIFFERENTIAL POLYNOMIALS

In [8] Hayman's problem is related to the $f'-af^{n}\neq b$ where $a,b\in C$ and $a\neq 0$. Hayman [8] Demonstrated meromorphic function on C that satisfies $f'-af^{n}\neq b$

Should be continuous, if $n\geq 5$. If f is entire then the result is' true for $n\geq 3$. For analytic functions, the normality result corresponding to s Hayman's theorem was proved by Drasin D[3], The corresponding result for meromorphic functions $(n\geq 5)$ was established (independently) by Langley.J.K, Xianjin Li etc. Now, we prove the following theorem using Zalcman lemma.

Here, we take, $\Psi(f)=M_s(f)+\sum_{j=1}^{s-1}a_jM_j(f)$ where a_j's are small holomorphic functions (unless otherwise stated) and each $M_j(f)=(f)^{n_{0j}}(f')^{n_{1j}}...(f^{(k)})^{n_{kj}}$ is a monomial generated by f. As usual $\gamma_j=\sum_{i=0}^{k}n_{ij}$ and $\Gamma_j=\sum_{i=0}^{k}(i+1)n_{ij}$ denote degree and weight of $M_j(f)$, respectively. We also take, $M_s(f)$ as the term with highest degree and weight among $M_j(f)$ $(j=1,2,\ldots s)$ and hence $\Gamma_\Psi=\Gamma_{M_s}$ and $\gamma_\Psi=\gamma_s$.

We need the following lemmas for further debate.

Lemma: Let f be a meromorphic function not constant in the complex plane which has only poles of order at least p, a be a non-zero finite complex number n, k, p, $\in$ N with $n\geq\Gamma_\Psi+1$ and also $n_{1s}\geq 1$, $n_{ij}\geq 0\ \forall i,j$. Suppose that f is not a polynomial of degree less than k, then

$$\left\{n-\gamma_s-\frac{(n_{1s}+2n_{2s}+....+kn_{ks})}{p}\right\}T(r,f)\leq\frac{1}{p}N(r,f)+N\left(r,\frac{1}{f}\right)$$

$$+\overline{N}\left(r,\frac{1}{M_s(f)-af^{n}}\right)+S(r,f)$$

Where, $S(r,f)=0\,(T(r,f))$ as $r\to\infty$ possibly outside a set of finite linear measure.

Proof: It is easy to see that $M_s(f)-af^{n}\neq 0$

NORMALITY CRITERIA FOR A FAMILY OF MEROMOPHIC FUNCTIONS AND HOMOGENEOUS DIFFERENTIAL POLYNOMIALS

Let f be a meromorphic function. Let us define a monomial in f, by $M_j(f(z))=(f)^{n_{0j}}(f')^{n_{1j}}\ldots(f^{(k)})^{n_{kj}}$ where $n_{0j}, n_{1j},\ldots n_{kj}$ are all positive integers. We call $\gamma_{Mj}=\sum_{i=0}^{k}n_{ij}$ the degree and $\Gamma_{M_j}=\sum_{i=0}^{k}(i+1)n_{ij}$ the weight of the monomial $M_j(f(z))$. Let $\Psi(f(z))=\sum_{j=1}^{s}a_jM_j(f(z))$ where a_j's are constants and for convince we write $\Psi(f(z))=M_s(f(z))+\sum_{j=1}^{s-1}a_jM_j(f(z))$ Then $\Psi(f(z))$ Is called polynomial differential with degree in f $\gamma_\Psi=\max_{1\leq j\leq s}\gamma_{Mj}$ and weight $\Gamma_\Psi=\max_{1\leq j\leq s}\Gamma_{M_j}$. condition $\gamma_\Psi=\gamma_{Mj}$ and $\Gamma_\Psi=\Gamma_{Mj}$ for $j=1,2,\ldots s$ then $\Psi(f(z))$ Is called the homogeneous polynomial derivative in f.

The concerns Hayman has about normal families are all of similar form. In each case, it is understood that a property concerning the values of a function and its derivatives means that a whole or globally defined meromorphic function has to be constant. Does a family of meromorphic functions have the same property imply normality? Hayman proved that an entire or a meromorphic function which satisfies $f\neq 0, f^{(n)}\neq 1$ for fixed $n\geq 1$ must be a constant. The corresponding results on normal families were proved by Yang Lo. Recently Ming-Liang Fang and W. Hong proved the following theorem:

Theorem A: Let F be a family of meromorphic functions in a domain D $k, q\ (\geq 2)$ be two positive integers, $\Psi(f(z)) = \sum_{j=1}^{s} a_j M_j(f(z))$ be a differential polynomial in f and $\frac{\Gamma_\Psi}{\gamma_\Psi} < k+1)$. If the zeros of f(z) are of multiplicity $\geq (k+1)$ and $(f^{(k)})^q + \Psi(f(z)) \neq 1$ for each $f \in F$, then F is normal in D.

Now we prove the following two theorems.

Theorem: Let F be a family of meromorphic functions in a domain D such that each $f \in F$ satisfies $f(z) \neq 0, \Psi(f(z)) \neq 1$ for any $z \in D$ then F is normal in D, where $\Psi(f(z)) = M_s(f(z)) + \sum_{j=1}^{s-1} a_j M_j(f(z))$ is a homogeneous differential polynomial and $\gamma_\Psi = \gamma_{Ms}$.

For the proof we need the following important lemmas.

As an application of Theorem we deduce the following Lemma

Lemma: Suppose f(z) is a transcendental meromorphic function in C, $f(0) \neq 0, \infty$. $M_j(f(z))$ Is a monomial in f ' and not identically constant then f(z) infinitely always assumes any finite value or $M_j(f(z))$ Assumes infinitely always every finite, non-zero value.

Proof of Theorem: Denote F to the family in question, and assume F is not natural. One may presume as normal $D = \{z;\ |z|<1\}$ select $\alpha = \frac{-(\Gamma_{Ms} - \gamma_{Ms})}{\gamma_{Ms}}$ By Lemma, It exists $f_j \in F,\ z_j \in D$ A list of positive numbers $\rho_j \to 0$ So that $\rho_j^{\alpha} f_j(z_j + \rho_j \xi) = g_j(\xi)$ converges to $g(\xi)$ locally - uniformly where g is a non-constant meromorphic function in D. Since $g_j \neq 0$ and g is a non-constant $g \neq 0$ by Hurwitz's theorem. Also $M_s(g_j(\xi))$ is the uniform limit of

$$M_s(g_j(\xi)) + \sum_{j=1}^{s-1} \rho_j^{\beta_j} a_j(z_j + \rho_j \xi) M_j(g_j(\xi)) \qquad \ldots(1)$$

Where

$\beta_j = \Gamma_{Ms} - \Gamma_{Mj} \geq 0$

$$= \rho_j^{\left(\frac{\Gamma_{Ms} - \gamma_{Ms}}{\gamma_{Ms}}\right)\gamma_{Ms} + \Gamma_{Ms} - \gamma_{Ms}} M_s(f_j(z_j + \rho_j \xi)) + \sum_{j=1}^{s-1} \rho_j^{\beta_j + \alpha\gamma_{Ms} + \Gamma_{Ms} - \gamma_{Ms}} a_j(z_j + \rho_j \xi) M_j(f_j(z_j + \rho_j \xi))$$

$$= M_s(f_j(z_j + \rho_j \xi)) + \sum_{j=1}^{s-1} a_j(z_j + \rho_j \xi) M_j(f_j(z_j + \rho_j \xi))$$

$$= \Psi((f_j(z_j + \rho_j \xi))) \neq 1$$

By hypothesis.

Thus $M_s(g_j(\xi)) \neq 1$ and $M_s(g_j(\xi))$ converges to $M_s(g(\xi))$ imply, either $M_s(g(\xi)) \neq 1$ or $M_s(g(\xi)) \equiv 1$.

Case: If $M_s(g(\xi)) \equiv 1$ then $(g^{(i)}(\xi)) \equiv 1$ for all $i = 0, 1, \ldots, k,$ which is impossible.

Case: If $M_s(g(\xi)) \neq 1$ then we arrive at a contradiction to Lemma. Consequently F is natural in D.

Theorem: Let F be a meromorphic functional family having zeros of order at least k, in a domain in D. Suppose that there exists a constant $M > 0$ such that, for each $f \in F$.

$$|f(z)| > M \qquad \ldots(2)$$

Whenever $z \in E(f) = \{z; \Psi(f(z)) = 1\}$. then F is normal in D.

Proof: Let $z_0 \in D$ and take $\alpha = \frac{-(\Gamma_{Ms} - \gamma_{Ms})}{\gamma_{Ms}}$. obviously $\alpha < 0$. When F isn't normal at z_0 Instead, there is a series of Lemma $g_j(\xi) = \rho_j^{\alpha} f_j(z_j + \rho_j \xi)$ Positive figures $\rho_j \to 0$ and a sequence Z_j such that $g_j(\xi)$ converges to a non-constant meromorphic function $g(\xi)$ spherically and locally uniformly in D. Also since g is not a polynomial of degree k, if g is a rational function or transcendental function then $M_s(g(\xi)) - 1$ also has zeros. Thus there exists a, ξ_0 such that

$$M_s(g(\xi_0)) = 1. \qquad \ldots(3)$$

Then for large m and $\xi \in \{|\xi - \xi_0| < 1/m\}$, there exists a positive constant $\varepsilon_0 > 0$, such that $|M_s(g(\xi)) - 1| \geq \varepsilon_0$

On the other hand, there exists a natural number $N > 0$ such that for $j \geq N$

$$|M_s(g_j(\xi)) - M_s(g(\xi))| < \varepsilon_0.$$

But it is the uniform limit of

$$M_s(g_j(\xi)) + \sum_{i=1}^{s-1} \rho_j^{\beta_i} a_i(z_j + \rho_j\xi) M_i(g_j(\xi))$$

$$= M_s(f_j(z_j + \rho_j\xi)) + \sum_{j=1}^{s-1} a_i(z_j + \rho_j\xi) M_i(f_j(z_j + \rho_j\xi))$$

$$= \Psi((f_j(z_j + \rho_j\xi)))$$

Hence, we have,

Thus, by Rouche's theorem, there exists a point $\xi_j \in \{|\xi_j - \xi_o| < 1/m\}$, such that $\Psi((f_j(z_j + \rho_j\xi))) = 1$

Combining this with the hypothesis, we have

$$\rho_j^{-\left(\frac{\Gamma_{Ms} - \gamma_{Ms}}{\gamma_{Ms}}\right)} |f_j(z_j + \rho_j\xi)| \geq M\rho_j^{-\left(\frac{\Gamma_{Ms} - \gamma_{Ms}}{\gamma_{Ms}}\right)} \quad \ldots(4)$$

Without loss of generality, we suppose that $\xi_j \to \xi_o'$; then $|\xi_o' - \xi_o| < 2/m$. Now the left hand side of (4) converges to $g(\xi_o')$ and the right side of (4) tends to ∞, by the fact that $\rho_j \to 0$ we have $g(\xi_o') = \infty$. Let $m \to \infty$ then $\xi_o' = \xi_o$ Thus $g(\xi_o) = \infty$ which contradicts (3). Hence the proof.

REFERENCES

Cherry W. and Ye Z. (2001). Nevanlinnas throry of Value Distirbution. Springer Verlag, Berlin.

Fang M. L. and W. Hong (2000). Unicity theorems for entire funnctions concerning diffemtial polynomials .Indian. Jou. of Pure and App. Math 32(9), pp. 1343-1348.

Fang M.L and W. Hong (2001). A unicity theorem for entire functions concerning differential polynomials, Indian Journal of Pure and Appl. Mathematics. 32(9) pp. 1343-1348.

Gol'dberg A.A and Ostrovskii I.V. (1970). The Distribution of Values of Meromorphic functions[in Russian].Nauki, Moscow.

Mues. E.and Steinmetz N. 1979). Meromorphe Funktionen, die mit ihrer Ableitung Werte teilen, Manuscripta Math., 29, pp. 195-206.

Nevanlinna. R. (1970). Analytic Functions, Springer Verlag, New York.

Puranik, A.G. (2002). Ph.D. Thesis, Karnatak University, Dharwad.

Rainer Bruck (1996). On entire functions which share one value CM with then ins' derivative, Results in Math., 30, pp. 21-24.

S.M. Sarangi and S.J. Patil. (1979). On Modified Deficiencies of Meromorphu Functions -Indian J.Pure appl. Math, 10(1): pp. 6-13.

Singh S.K. and Kulkarni V.N. (1973). Characteristic function of meromorphk function and its derivative; Annales Polonici Mathematic xxviii. pp. 123-133.

Corresponding Author

Kate Sunil Krishnaji*

PhD Student, MUIT, Lucknow