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A Note on Lifting of Idempotents

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Abstract – Let I be an ideal of a ring R with identity. We say that idempotents modulo I can be lifted if for every a $\in$ R such that a^2 – a $\in$ I there exists an idempotent e $\in$ R such that e – a $\in$ I. In this review, we will discuss various results about lifting idempotents and characterization of rings in terms of this property.

Keywords – Exchange Ring, Indecomposable Ideal, Morphic Ring, Regular Elements, Stably Free Projective Module, Suitable Ring, Semilocal Ring

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1. INTRODUCTION

In Algebra, it is sometimes useful to construct new algebraic structures from the existing algebraic structures. Often these new structures turn out to be simpler and from these simple structures we can go back to the original structure using some known properties. One such example of this process is studying the factor rings of a ring *R* and then lifting the properties of the factor ring back to *R*. The study of lifting idempotents modulo ideals is an important tool to study the structure of rings in this direction. Many people have studied the lifting of idempotents modulo a one-sided ideal *R*. For instance, It is a classical result that if *R* is a nil ideal then idempotents can be lifted. Koh [1] gave an elementary proof of this result and also proved that the result is true even for the rings without identity.

In literature some classes of rings are even defined in terms of lifting properties. For example, a potent ring *R* is a ring whose idempotents lift modulo Jacobson radical and every left (equivalently right) ideal which is not contained in the Jacobson Radical contains a non-zero idempotent. Several classes of rings are also characterized by the lifting idempotents modulo ideals.

2. LIFTING IDEMPOTENTS

A right *R*−module *M* is said to have the exchange property (see Crawley and Jónsson [2]) if for any module *X* and decompositions $X = M' \oplus Y = \oplus_{i \in I} N_i$ where $M' \simeq M$, there exists submodules $N'_i \subseteq N_i$ for each i such that $X = M' \oplus (\oplus_{i \in I} N'_i)$, and a module M_R has the finite exchange property if the above condition is met whenever the indexing set *I* is finite. Warfield in [3] called a ring *R* an exchange ring, if R_R has the finite exchange property, equivalently, if it has the full exchange property. He proved that that the definition of an exchange ring is left-right symmetric. Local rings, semi regular rings and von Neumann regular rings are all the examples of exchange rings.

Nicholson [4] in his classical paper proved that Exchange rings of Warfield can be characterized by lifting properties of idempotents. For this purpose, Nicholson introduced suitable rings.

A ring *R* is called suitable ring if for each $x \in$ *R*, $\exists\ e^2 = e \in Rx$ such that $1 - e \in R(1 - x)$.

Nicholson further proved that a ring is suitable ring if idempotents can be lifted modulo every left ideal.

Proposition 1.1 (Nicholson [4]) A ring *R* is suitable iff *R/J(R)* is suitable and idempotents can be lifted modulo *J(R)*.

As for a ring *R* with identity, end ${}_RR \cong R$, the following important result by Nicholson proves that Exchange rings are same as suitable rings and hence Exchange rings can be characterized by lifting idempotents modulo ideals.

Theorem 1.2 (Nicholson [4]) If *R* is a ring, the following conditions are equivalent for a left *R*-module *M*:

(1) End M is right suitable,

(2) M has the finite exchange property,

(3) End M is left suitable.

Let *R* be a ring such that the idempotents lift modulo *J(R)*, where *J(R)* denotes the Jacobson Radical of ring *R*. Then $a^2 - a \in J(R)$ implies that there exists $e^2 = e \in aR$ such that $e - a \in J(R)$ (see Lemma 5 [5]).

Nicholson and Zhou [5] called this property to be strong *lifting property* of idempotents. If *I* is an ideal of a ring *R*, we say that idempotents can be lifted strongly modulo *I* if, whenever $a^2 - a \in I$, there exists $e^2 - e \in aR$ such that $e - a \in I$. In the following lemma, they proved that this condition is left-right symmetric even in the case of one-sided ideal.

Lemma 1.3 (Nicholson, Zhou [5]) *The following are equivalent for a right ideal T of R:*

(1) *If $a^2 - a \in T$, there exists $e^2 = e \in aR$ such that $e - a \in T$.*

(2) *If $a^2 - a \in T$, there exists $e^2 = e \in aRa$ such that $e - a \in T$.*

(3) *If $a^2 - a \in T$, there exists $e^2 = e \in aRa$ such that $e - a \in T$.*

Even for a commutative ring, Strong lifting is much stronger than ordinary lifting. For example [2] if $R = \mathbb{Z}$ and $I = p^k\mathbb{Z}$, where p is a prime number and *k* is an integer greater than equal to 1, then clearly as the only idempotents in *R/I* are 0 and 1, idempotents lift modulo *I*. Taking $b = p^k$ and $1 - p^k$, we get $a^2 - a = b^2 - b \in I$. Now 0 cannot lift a, as $0 - a \notin I$. So clearly 1 lifts the idempotent and this is not strong lifting as $1 \notin aR$.

However, in some cases ordinary lifting can imply strong lifting. The following result is true in this direction.

Lemma 1.4 (Nicholson, Zhou [5]) *If idempotents can be lifted in R modulo J, they can be lifted strongly modulo every one-sided ideal contained in J.*

Strengthening Nicholson's result, they proved that exchange rings have in fact strong lifting property modulo each left and right ideal of the ring.

Theorem 1.5 [Nicholson, Zhou [5]] *A ring R is an exchange ring if and only if every right (respectively left) ideal of R is strongly lifting.*

As already discussed, idempotents may not be lifted modulo an arbitrary ideal and even if they are lifted, lifting may not be strong lifting. However, Menal [6] proved the following result:

Proposition 1.6 (Menal, [6]) *Every π −regular right ideal (and hence every nil right ideal) is strongly lifting.*

Further Zhu [7] in 2008 investigated the problem of lifting of central idempotents. An idempotent *e* is called central if and only if $eRf = fRe = 0$, where $f = 1 - e$ is the complementary idempotent of *e* and we say that a central idempotent $\bar{g} \in R/I$ can be lifted to *R*, if there exists a central idempotent $e \in R$ such that $e - g \in I$. For a ring *R*, a finitely generated projective *R*-module P is called *stably free* if $P \oplus R^m$ is free for some integer *m*. P is called power *stably free* if P^m is *stably free* for some positive integer *m*.

Zhu associated the problem of lifting central idempotents of a Noetherian (artinian) ring with the rank of K_0 groups. For an abelian group G, the rank of G, denoted by *rank*(G) is defined as the dimension of $G \otimes \mathbb{Q}$ considered as a vector space over $\mathbb{Q}$.

An ideal *J* of a ring *R* is called *indecomposable* if *J* is not a direct sum of two nonzero ideals of *R*. *J* is called a block ideal of *R* if *J* is a direct summand of *R* and *J* is indecomposable.

$$BI(R) = \{J \mid J \text{ is a block ideal of R and has IBN}\}$$

A finitely generated projective R − module P is called *relative power stably free* if for each $R_i \in BI(R)$, $P \otimes R_i$ is power stably free as an R_i − module and $P \otimes R_c$ is power stably free as an R_0 − module.

He proved the following suitable condition relative to the rank of K_0 groups.

Theorem 1.7 (Zhu, [7]) *Let R be a noetherian (artinian) ring. If each finitely generated projective R-module is relative power stably free and rank* $(K_0(R)$ *= rank* $K_0(R/J(R))$*, then the central idempotents can be lifted modulo J(R).*

A ring *R* is called *semilocal* if *R/J(R)* is a left artinian ring. In the following result, Zhu proved that for semilocal rings, even the converse of above theorem is true.

Theorem 1.8 (Zhu, [7]) *Given a semilocal ring R, the following statements are equivalent:*

(1) *Each central idempotent in R/J(R) can be lifted to a central idempotent in R;*

(2) *rank* $(K_0(R) = \text{rank}\ K_0(R/J(R))$ *and each finitely generated projective R-module is relative power stably free.*

In this case, $K_0(R) \cong K_0(R/J(R))$

If *R* is a ring, an element *a* in *R* is called *left morphic* if $R/Ra \cong l(a)$ where $l(a)$ denotes the left annihilator of *a* in *R*. The ring itself is called *a left morphic ring* if its every element is left morphic.

The following result is proved by Zhu for one-sided morphic rings.

Proposition 1.9 (Zhu, [7]) *Let R be a left morphic ring. If the idempotents in R/J(R) can be lifted to the idempotents in R, then the central idempotents in R/J(R) can be lifted to the central idempotents in R.*

Nicholson proved that *R* is an exchange ring if and only if the *n-by-n* full matrix ring $Mat_n(R)$ over *R* is an exchange ring.

The following results about extensions of Exchange rings are proved by Hong at el. In [8]

Proposition 1.10 (Hong, Kim, Lee, [8]) *The following statements are equivalent:*

(1) R is an exchange ring.

(2) Every upper triangular matrix ring (finite or column finite) over R is an exchange ring.

(3) Every lower triangular matrix ring (finite or row finite) over R is an exchange ring.

Proposition 1.11 (Hong, Kim, Lee, [8]) *Let R be an exchange ring. Then* $\frac{R[x]}{<x^n+1>}$ *is an exchange ring for any* $n \geq 1$.

Theorem 1.12 (Hong, Kim, Lee, [8]) *Let* α *be an endomorphism of R. Then the following statements are equivalent:*

(1) R is an exchange ring.

(2) The formal power series ring R[[x]] of R is an exchange ring.

(3) The skew power series ring R[[x; a]] of R is an exchange ring.

And so it follows that property of lifting idempotents modulo every left ideal passes from a ring *R* to upper (lower) triangular matrix rings, formal power series ring *R[[x]]* and skew-power series ring *R[[x; a]]* for some endomorphism *a* of *R*.

But, in general this property does not pass to subrings. In other words, we can say that subring of an exchange ring is not exchange. For example, $\mathbb{Q}$, the field of all rational numbers, is an exchange ring but the subring $\mathbb{Z}$, the integer of integers, is not exchange. Petar Pavešić, using the idea of induced lifting, proved that under certain assumptions, subring of an exchange ring is also an exchange ring.

Theorem 1.13 (Pavešić, [9]) *Let R be a subring of an exchange ring* E *and assume that one of the following conditions is satisfied (*R^0*denotes the set of all non-invertible elements of R):*

(a) R is noetherian and $E \cdot I \subseteq R$ *for some left ideal I of finite index in R;*

(b) For every $x \in R^0$*, there exists an integer n such that* $E \cdot x^n \subset R$*;*

(c) R is semi complete and for every $x \in \bigcap_k (R^0)^k$*, there exists an integer n such that* $E.x^n \subset R$.

Then *R* is an exchange ring.

A ring-theoretic property *P* is called *Morita-invariant* if, whenever *a* ring *R* has the property *P*, then any ring *S* Morita-equivalent to *R* also does. Diesl et. al. in[10] showed that property of lifting idempotents modulo a left ideal is not Morita invariant by proving the following result:

Theorem 1.13 (Diesl, Dittmer, Nielsen, [10]) *If idempotents lift modulo the Jacobson radical J(S) of some ring S, idempotents do not necessarily lift modulo* $J(M_2(S))$ *in the ring* $M_2(S)$.

Several authors have explicitly found out the idempotents in a particular ring. For example, Meenu Khatkar [11] found out the idempotents in $M_2(\mathbb{Z}_6[x])$ and proved that

Theorem 1.15 (Khurana [11]) *Any non-trivial idempotent in* $M_2(\mathbb{Z}_6[x])$ *is one of the following form*

1. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$

2. $\begin{bmatrix} a(x) & b(x) \\ c(x) & 1-a(x) \end{bmatrix}$, where $a(x)(1-a(x))-b(x)c(x) = 0$

3. $\begin{pmatrix} 3a(x) & 3b(x) \\ 3c(x) & 3(1-a(x)) \end{pmatrix}$, where $a(x)(1-a(x)-b(x)c(x)=2f(x)$

4. $\begin{pmatrix} 2a(x) & 2b(x) \\ 2c(x) & 4-2a(x) \end{pmatrix}$, where $a(x)(1-2a(x))-2b(x)c(x) = 3g(x)$

5. $\begin{pmatrix} 3+2a(x) & 2b(x) \\ 2c(x) & 1-2a(x) \end{pmatrix}$, where $a(x)(1-2a(x))-2b(x)c(x) = 3h(x)$

6. $\begin{pmatrix} 1+3a(x) & 3b(x) \\ 3c(x) & 4-3a(x) \end{pmatrix}$, where $a(x)(1-ax-bxcx=2\phi(x)$

Where a(x), b(x), c(x), f(x), g(x), h(x), ϕ*(x) are polynomials in* $\mathbb{Z}_6[x]$*. Non-zero*

CONCLUSION:

The problem of lifting idempotents modulo ideals has been studied by many authors in literature and many rings are defined and characterized in terms of lifting properties. In this article, we review the various results in literature about lifting of idempotents and characterization of rings in terms of this property.

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