Generating Relations Involving Hypergeometric Function by Means of Integral Operators

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Abstract – The nucleus of excavation is based on the results which involve exponential function. The results of Exton and Pathan & Yasmeen are used with a view to obtain multivariable generating functions which are partly bilateral and partly unilateral.

Keywords: Integral representation, Lagurre Polynomials, Hypergeometric Function, Generating Functions, Variables, Function.

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1. INTRODUCTION

It should be noticed that the exponential function, which is a special case of the generalized hypergeometric function pFq and p = q = 0, appears in many different situations; for instance, in conformal mapping theory [151], in automorphic function theory [151], in the theory of representation of Lie algebras, in Physics and in the theory of differential equations [87]. The leading example of partly bilateral and partly unilateral generating function in terms of exponential function is doubtless the result of Exton ([38]; p. 147(3)]

$$\exp\left(s+t-\frac{xt}{s}\right) = \sum_{M=-\infty}^{\infty}\sum_{N=0}^{\infty} s^M t^N F_N^M(x), \qquad (1.1)$$

Where

$$F_N^M(x) = \frac{{}_1F_1[-N;M+1;x]}{M!N!} = \frac{L_N^{(M)}(x)}{(M+N)!} \qquad (1.2)$$

and $F_N^M(x)$ are the classical Laguerre polynomials ([121; p. 200(1)).

Pathan and Yasmeen [109] modified Exton's result (1.1) by defining $M^* = \max(0,-M)$ and

$$F_N^M(x) = \frac{L_N^{(M)}(x)}{(M+N)!} = \frac{1}{N!}\sum_{r=M^*}^{N}\frac{(-N)_r x^r}{(M+r)!r!}, \text{ if } N \geq M$$

$$= 0, \text{ if } 0 \leq N < M^* \quad (\text{i.e., if } M+N<0\leq N)$$

So that all factorials of negative integers occurring in this definition have meaning. Thus we may rewrite equation (1.1) in the form

$$\exp\left(s+t-\frac{xt}{s}\right) = \sum_{M=-\infty}^{\infty}\sum_{N=0}^{\infty}\frac{s^M t^N}{(M+N)!}L_N^M(x). \qquad (1.3)$$

This result has attracted a great deal of interest by several researchers including (for example) Pathan and Yasmeen ([107], [108] and [109]). Goyal and Gupta [52], Srivastava et al. ([142] and [143]). Gupta et al [60], Kamarujjama et al. [79], and Pathan and Subuhi [116]. Works on Exton's result inspired us to use integral operators to obtain more generating functions, which are partly bilateral and partly unilateral.

On account of many properties of generating functions which are partly bilateral and partly unilateral, an increasing, number of such problems and properties are now capable of being elegantly represented by their use. A number of such, generating functions are obtained in this paper.

In section 1.2, an operator □ is explored and some generating functions are obtained by making use of results of Exton [38] and Pathan and Yasmeen [109]. Further, a number of multiple seri4s of hypergeometric functions are obtained in Section 1.3.

2. INTEGRAL OPERATOR Ω AND GENERATING FUNCTIONS:

If we define the integral operator Ω by

$$\Omega_{\lambda,\mu}\{\ \} = \int_0^1 x^{\lambda-1}(1-x)^{\mu-1}\{\}dx$$

Then rewriting the results ([130] p. 36(6)), ([45]; p.192(50), with y = 1), ([45]; p.193 (51), with y = 1) in terms of this operator, we have the results

$$\Omega_{\lambda,\mu}\{e^{zx}\} = \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}\,{}_1F_1\left[{\lambda; \atop \mu;}z\right], \operatorname{Re}(\mu)>\operatorname{Re}(\lambda)>0 \qquad (2.1)$$

$$\Omega_{\lambda,\mu}\{L_n^{(\alpha)}(\beta x)\} = \frac{\Gamma(\lambda)\Gamma(\mu)(\alpha+1)_n}{n!\Gamma(\lambda+\mu)}\,{}_2F_2\left[{-n,\lambda; \atop \alpha+1,\lambda+\mu;}\beta\right], \operatorname{Re}(\lambda),\operatorname{Re}(\mu)>0 \qquad (2.2)$$

And

$$\Omega_{\lambda,\mu}\{L_n^{(\alpha)}(\beta x)\} = \frac{\Gamma(\lambda)\Gamma(\mu)(\alpha+1)_n}{n!\Gamma(\lambda+\mu)}\,{}_2F_2\left[{\alpha+n+1,\lambda; \atop \alpha+1,\lambda+\mu;}-\beta\right], \operatorname{Re}(\lambda)\cdot\operatorname{Re}(\mu)>0 \qquad (2.3)$$

Starting from the result (2.1) with z replaced $s+t-\frac{yt}{s}$, we have

$$\Omega_{\lambda,\mu}\left\{\exp\left(s+t-\frac{yt}{s}\right)x\right\} = \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{n!\Gamma(\mu)}\,{}_1F_1\left[{\lambda; \atop \mu;}s+t-\frac{yt}{s}\right], \operatorname{Re}(\mu)>\operatorname{Re}(\lambda)>0. \qquad (2.4)$$

Now using the results (1.3), (2.1) and (2.2), we can establish the following result

$${}_1F_1\left[{\lambda; \atop \mu;}s+t-\frac{yt}{s}\right] = \sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\frac{(\lambda)_{M+N}s^{M}t^{N}}{(\mu)_{M+N}M!N!}\,{}_2F_2\left[{-N,M+N+\lambda; \atop M+1,M+N+\mu;}y\right], \operatorname{Re}(\lambda), \operatorname{Re}(\mu)>0 \quad (2.5)$$

Further letting $s=t=\frac{y}{2}$, one has the following closure relation

$$1 = \sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\frac{\left(\frac{y}{2}\right)^{M+N}(\lambda)_{M+N}}{(\mu)_{M+N}M!N!}\,{}_2F_2\left[{-N,M+N+\lambda; \atop M+1,M+N+\mu;}y\right], \operatorname{Re}(\lambda),\operatorname{Re}(\mu)>0 \qquad (2.6)$$

which for $\lambda=\mu$, can be written as,

$$1 = \sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\frac{\left(\frac{y}{2}\right)^{M+N}}{(\mu)_{M+N}M!N!}\,{}_1F_1\left[{-N; \atop M+1;}y\right] \qquad (2.7)$$

Again taking in (2.1) for $z=s+t-y-\frac{yt}{s}$, we have

$$\Omega_{\lambda,\mu}\left\{\exp\left(s+t-\frac{yt}{s}\right)x\right\} = \frac{\Gamma(\lambda)\Gamma(\mu-\lambda)}{\Gamma(\mu)}\,{}_1F_1\left[{\lambda; \atop \mu;}s+t-y-\frac{yt}{s}\right], \operatorname{Re}(\mu)>\operatorname{Re}(\lambda)>0.$$

Now using the above result, together with the results (2.3) and (1.3), we are led to

$${}_1F_1\left[{\lambda; \atop \mu;}s+t-y-\frac{yt}{s}\right] = \sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\frac{(\lambda)_{M+N}s^{M}t^{N}}{(\mu)_{M+N}M!N!}\,{}_2F_2\left[{M+N+1,M+N+\lambda; \atop M+1,M+N+\mu;}-y\right], \operatorname{Re}(\lambda),\operatorname{Re}(\mu)>0 \quad (2.8)$$

Which for □ = μ , yields

$$\exp\left(s+t-y-\frac{yt}{s}\right) = \sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\frac{s^{M}t^{N}}{M!N!}\,{}_2F_2\left[{M+N+1 \atop M+1}-y\right] \qquad (2.9)$$

For $s=t=\frac{y}{2}$, equation (2.8) reduce to

$${}_1F_1\left[{\lambda; \atop \mu;}-y\right] = \sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\frac{(\lambda)_{M+N}\left(\frac{y}{2}\right)^{M+N}}{(\mu)_{M+N}M!N!}\,{}_2F_2\left[{M+N+1,M+N+\lambda; \atop M+1,M+N+\mu;}-y\right], \operatorname{Re}(\lambda),\operatorname{Re}(\mu)>0 \qquad (2.10)$$

3. GENERATING FUNCTIONS OF SEVERAL VARIABLES

The method of derivation of the generating functions involves the following results.

A result of Pathan ([106]; p. 52(5))

$$\int_0^\infty t^{\lambda-1}e^{-\left(z+\frac{r}{2}\right)l}W_{\lambda,\mu}(pt)\prod_{i=1}^{n}(M_{k,m,-1}(x_il))dt = \frac{\Gamma(A+\mu)\Gamma(A-\mu)p^{\mu+\frac{1}{2}}}{\Gamma\left(A-k+\frac{1}{2}\right)s^{l+\mu}}\prod_{i=1}^{n}(x_i^{m})$$

$$xF_p^{(n+1)}\left[\begin{matrix}A+\mu:A-\mu;m_1-k_1;m_2-k_2;\ldots m_n-k_n;\mu-k+\frac{1}{2}; \\ \frac{x_1}{\delta},\frac{x_2}{\delta},\ldots\ldots,\frac{x_n}{\delta},\frac{(\delta-p)}{\delta} \\ A-k+\frac{1}{2};-;2m_2;\quad 2m_2;\ldots\ldots;2m_n;\quad -;\end{matrix}\right], \qquad (3.1)$$

Where

$$A=\frac{1}{2}+\lambda+\sum_{i=1}^{n}m_i,\ \delta=z+p+\frac{1}{2}\sum_{i=1}^{n}x_j\ \operatorname{Re}(A+\mu)>0,$$

$\operatorname{Re}\left(2z+p-\sum_{i=1}^{n}x_i\pm p\pm\sum_{i=1}^{n}m_i\right)>0$, and $F_p^{(n+1)}$ is a generalized hypergeometric function of (n+1) variable ([106]; p. 51(1)),

$$F_p^{(n+1)}\left[{a:b;d_1;d_2;\ldots\ldots;d_n;d; \atop e:-;e_1;e_2;\ldots\ldots;e_n;-;}z_1,z_2,\ldots,z_n,z\right]$$

$$= \sum_{s_1,s_2,\ldots,s_n r=0}^{\infty}\frac{(a)_{r+s_1,s_2,\ldots,s_n}(b)_{s_1,s_2,\ldots,s_n}(d)_r\prod_{k=1}^{n}\left((d_k)_{a_k}(z_k)^{N_a}\right)z^r}{(c)_{r_1+s_1,s_2,\ldots,s_n}\prod_{k=1}^{n}\left((e_k)_{s_k}(s_k)!\right)r!}, \quad (3.2)$$

Where $|z|,|z_k|<1, k\in\{1,2,\ldots,n\},(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}$ and by analytic continuation, none of the quantities c, e_1, e_2, … , e_n are zero or a negative integer.

The result of Pathan and Yasmeen ([108]; p.5(3.1))

$$L_{m_1}^{(a_1)}(x_1)L_{m_2}^{(a_2)}(x_2)L_{m_3}^{(a_3)}(x_3) = \frac{(1+a_1)_{m_1}(1+a_2)_{m_2}(1+a_3)_{m_3}}{m_1!m_2!m_3!}$$

$$\times\sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\frac{(-m_1)_M(-m_2)_N x_1^{M}x_2^{N}}{(1+a_1)_M(1+a_2)_N M!N!}$$

$$X_4F_4\left[{-m_1+M,-a_2-N,-m_3,-N \atop 1-a_1+M,1+m_2-N,1+a_3,M+1;}\ \frac{-x_1x_3}{x_2}\right] \qquad (3.3)$$

Expressing Laguerre polynomial $L_m^{(a)}(x)$ in terms of confluent hypergeometric function ${}_1F_1$ using (1.2) and further using the relation between and the Whittaker's function of the first kind $M_{k,\mu}$([130];p.39(23))

$$M_{k,\mu}(z) = z^{\mu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) {}_1F_1\left[\mu - k + \frac{1}{2}; 2\mu + 1; z\right],$$

We see that the Laguerre polynomial $L_m^{(a)}(x)$ is related to the Whittaker function $M_{k,\mu}(x)$ by the equation

$$L_m^{(a)}(x)\left(\frac{(m+a)!}{m!a!}\right)\exp\left(\frac{x}{2}\right)x^{\frac{n+1}{2}} M_{\frac{1}{2}(a+1)m,\frac{a}{x}}x. \qquad (3.4)$$

Now, replacing x_1, x_2 ad x_3 by x_1u, x_2u and x_3u, respectively in (3.3) and multiplying both the sides by

$$\prod_{i=4}^{n} L_m^{(a)}(x_1 u)\exp\left(-\left(z+\frac{p}{2}\right)u\right)M_{k,j_i}(pu),$$

Further using (3.4) to replace each of the Laguerre polynomials, and finally integrating with respect to u from zero to infinity, we arrive at

$$\prod_{i=1}^{n}\left((x_1)\frac{a_1+1}{2}\right)\int_0^\infty u^{\frac{-1}{2}\left(\Sigma_{-1}^{\infty}a_1 4n\right)_x + \left(z+\frac{1}{2}\left(p-\Sigma_{-1}^{\infty}a_1\right)\right)u} W_{k,\mu}\prod_{i=1}^{n} M_{\frac{1}{2}(a_1+1)+m_1,\frac{a_1}{2}}(x_1u)du$$

$$\prod_{i=4}^{n}\left((x_1)\frac{a_1+1}{2}\right)\times\sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\sum_{S=0}^{\infty}\frac{(-m_1)_M(-m_2)_N(-m_1+M)_S}{(1+a_1)_M(1+a_2)_N(1+a_2)_N(1+a_1+M)_S}$$

$$\times\frac{(-a_2-N)_S(-m_3)_S(-N)_S x_1^M x_2^N\left(-\frac{x_1x_3}{x_2}\right)^S}{(1-m_2-N)_S(1+a_3)_S(M+1)_S M!N!s!}$$

$$\times\int_0^\infty u^{M+N+S-\frac{1}{2}\left(\Sigma_{n=1}^{s}a_1+n-3\right)-\frac{1}{2}\left(z+\frac{1}{2}\left(p-\Sigma_{n=1}^{s}x_1\right)uW_{k,\mu}(pu)\right)}$$

$$\prod_{i=4}^{n}M_{\frac{1}{2}(a_1+1)+m_1,\frac{a_1}{2}}(x_1u)du \qquad (3.4)$$

Now using result (3.1), we get, after some simplifications

$$F_p^{(n+1)}\left[\begin{matrix}\frac{3}{2}+\mu;\frac{3}{2}-\mu;-m_1;m_2;\ldots;m_n;\mu-k+\frac{1}{2}; \\ \frac{x_1}{z+p},\frac{x_2}{z+p},\ldots,\frac{x_n}{z+p},\frac{z}{z+p} \\ 2-k;-;a_1+1;a_2+1;\ldots;a_n+1; \quad -;\end{matrix}\right]=\sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\sum_{S=0}^{\infty}\frac{(-m_1)_M(-m_2)_N(-m_1)_S(-a_2-N)_S(-m_3)_S}{(1+a_1)_M(1+a_2)_N(1+a_1+M)_S(1+m_2-N)_S(1+a_3)_S}$$

$$xF_p^{(n+1)}\left[\begin{matrix}\frac{3}{2}+\mu+M+N+s;\frac{3}{2}-\mu+M+N+s;-m_4;-m_5;\ldots;-m_n;\mu-k+\frac{1}{2}; \\ \frac{x_4}{z+p},\frac{x_5}{z+p},\ldots,\frac{x_n}{z+p}\text{ and }\frac{z}{z+p} \\ 2-k+M+N+s; \quad -; \quad a_4+1;\ a_5+1,\ldots a_n+1; \quad -;\end{matrix}\right], \qquad (3.6)$$

$$\mathrm{Re}\left(\frac{3}{4}\pm\mu\right)>0, \mathrm{Re}(z+p)>0, |z|,|x|<1, i=1,2,\ldots,n.$$

To make the above result more proper, we replace $\frac{x_1}{z+p},\frac{x_2}{z+p},\ldots,\frac{x_n}{z+p}$ and $\frac{z}{z+p}$ by $x_1,x_2,\ldots,x_n$ and z respectively and also $\frac{3}{2}+\mu$ and $2-k$ by a and b respectively to get

$$F_p^{(n+1)}\left[\begin{matrix}a:3-a;-m_1;-m_2;\ldots;a+b-3; \\ b:-a_1+1;\ a_2+1;\ldots;a_n+1;-;\end{matrix}\ x_1,x_2,\ldots,x_n,z\right]$$

$$=\sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\sum_{S=0}^{\infty}\frac{(-m_1)_M(-m_2)_N(-m_1+M)_S(-a_2-N)_S(-m_3)_S}{(1+a_1)_M(1+a_2)_N(1+a_1+M)_S(1+m_2-N)_S(1+a)_S}$$

$$\times\frac{(-N)_S(a)_{M+N+N}(3-a)_{M+N+S}x_1^M x_2^N\left(\frac{-x_1x_2}{x_2}\right)^S}{(M+1)_*(b)_{M+N+S}M!N!S!}$$

$$F_p^{(n-2)}\left[\begin{matrix}a+M+N+s:3-a+M+N+s;-m_4;-m_5;\ldots;-m_n;a+b-3; \\ x_4,x_5,\ldots,x_n,z \\ b+M+N+s:-;a_4+1;a_5+1;\ldots;a_n+1;-;\end{matrix}\right],\mathrm{Re}(a)>0,|z|,|x_i|<1,i=1,2,\ldots,n \qquad (3.7)$$

Special Cases:

1. Taking a = b and z □ 0 in equation (3.7) and redplacing 3-a by a,

We get

$$F_A^{(n)}[a,-m_1,-m_2,\ldots,-m_n,a_1+1,a_2+1,\ldots,a_n+1,x_1,x_2,\ldots,x_n]$$

$$=\sum_{M=-\infty}^{\infty}\sum_{N=M^*}^{\infty}\sum_{S=0}^{\infty}\frac{(-m_1)_M(-m_2)_N(-m_1+M)_S(-a_2-N)_S(-m_3)_S}{(1+a_1)_M(1+a_2)_N(1+a_1+M)_S(1+m_2-N)_S(1+a_3)_S}$$

$$\times\frac{(-N)_S(a)_{M+N+N}x_1^M x_2^N\left(\frac{-x_1x_3}{x_2}\right)^S}{(M+1)_s M!N!S!}$$

$$F_p^{(n-3)}[a+M+N+s,-m_4,-m_5,\ldots,-m_n,a_4+1,a_5+1,\ldots,a_n+1,x_4,x_5,\ldots,x_n]$$

$$|x_1|+|x_2|+\ldots+|x_n|<1 \qquad (3.8)$$

Where $F_A^{(n)}$ is Lauricella function [95] of n variables, defined as follows

$$F_A^{(n)}[a,b_1,b_2,\ldots,b_n,c_1,c_2,\ldots,c_n,x_1,x_2,\ldots,x_n]$$

$$=\sum_{m_1,m_2,\ldots,m_n=0}^{\infty}\frac{(a)_{m_1,m_2,\ldots,m_n}(b_1)_{m_1}(b_2)_{m_2}\ldots(b_n)_{m_n}(x_1)^{m_1}(x_2)^{m_2}\ldots(x_n)^{m_n}}{(c_1)_{m_1}(c_2)_{m_2}\ldots(c_n)_{m_n}m_1!m_2!\ldots m_n!}$$

$$|x_1|+|x_2|+\ldots+|x_n|<1 \qquad (3.9)$$

Further for n = 3, equation (3.8) reduces to

$$F_A^{(3)}[a,-m_1,-m_2,-m_3;a_1+1,a_2+1,a_3+1;x_1,x_2,x_3]$$

$$= \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \frac{(-m_1)_M(-m_2)_N(a)_{M+N}x_1^M x_2^N}{(1+a_1)_M(1+a_2)_N\, M!N!}$$

$$x_s F_4\left[\begin{matrix}-m_1+M,-a_2-N,-m_3,-N,a+M+N; & \frac{-x_1x_2}{x_2} \\ 1+a_1+M,\ 1+M_2-N;\ 1+a_3,\ M+1; & \end{matrix}\right] \qquad (3.10)$$

2. For n = 3 equivalently for $x_4 = x_5 = \ldots = x_n = 0$, equation (3.7) gives us

$$F_P^{(4)}\left[\begin{matrix}a::-a;-;-;-: & -m_1;-m_2;-m_3;a+b-3; & \\ b::-;-;-: & a_1+1;\ a_2+1;\ a_3+1;\ -; & \end{matrix} x_1,x_2,x_3,z\right]$$

$$= \sum_{M=-\infty}^{\infty} \sum_{N=M^*}^{\infty} \sum_{r=0}^{\infty} \frac{(-m_1)_M(-m_2)_N(a)_{M+N}(3-a)_{M+N}}{(1+a_1)_M(1+a_2)_N(b)_{M+N}} \times \frac{(a+M+N)_r(a+b-3)_r x_1^M x_2^N z^r}{(b+M+1)_r M!N!r!}$$

$$x_6F_5\left[\begin{matrix}-m_3-N,-m_1+M,a+M+N+r,3-a+M+N,-a_2-N; & \frac{-x_1x_3}{x_2} \\ 1+a_3,\ 1+a_1+M;\ 1+m_2-N,1+M,b+M+N+r; & \end{matrix}\right] \qquad (3.11)$$

Where $F_P^{(4)}$ is hyper geometric function of four variables considered by Pathan [8] and is defined as follows

$$F_P^{(4)}\left[\begin{matrix}a::b;-;-;-: & d;e;f;g; & \\ c::-;-;-: & d';e';f';-; & \end{matrix} u,x,y,z\right] = \sum_{m,n,p,q=0}^{\infty} \frac{(a)_{m+n+p+q}(b)_{m+n+p}(d)_m(e)_n(f)_p u^m x^n y^p z^q}{(c)_{m+n+p+q}(d')_m(e')_n(f')_p\, m!n!p!q!} \qquad (3.12)$$

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