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On Continuity of a Function Concerning Dynamics on Circle

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Abstract – In this paper we obtain some subsets of the set IR of real numbers on which fractional part function is continuous as a real-valued function of a real variable. Every real number can be written as the sum of its integral part and fractional part. This gives rise to fractional part function as a real-valued function of a real variable. This function is helpful in the study of dynamics on circle. Some properties of such subsets of IR are obtained.

Keywords – Dynamics on circle, Circle map, Fractional part function, Continuity, Separated sets.

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1. INTRODUCTION

Dynamics is the study of the motion of a body, or more generally evolution of a system with time (see e.g. [2], [3]). In dynamics on circle, motion takes place through the points of a circle. This movement of points is tried to be represented by a function defined on the circle. Such functions are called circle maps. The study of dynamics on circle also includes knowing about the continuity of the concerned functions. A circle in the plane IRxIR is the boundary of a disc with centre at a point of the plane and some radius. Corresponding to discs in IRxIR, there are spheres in IRxIRxIR, in fact, for every natural number n, there are spheres in IR^{n+1} (=IRxIRx…….IR(n+1) times). Thus the boundary of a sphere with centre at a point of IR^{n+1} and some radius can be considered as a 'circle' in IR^{n+1}. Since any two circles in IRxIR are homeomorphic, circle in the plane means the circle at the origin of the plane with unit radius for the study involving dynamics on circle. Such a circle is denoted by S^1. The corresponding circle in IR^{n+1} is denoted by S^n. But here we shall consider only S^1. A circle map is a function whose domain and codomain both are S^1. The continuity of a function at a point assures that the points which are close to the point of continuity (i.e. the points which are in a certain neighborhood of the points of continuity) are mapped, by the function, as close to each other as we want. Dynamics on the circle and dynamics of circle maps have been studied by many see e.g. [3], [4] and [5].

Every real number can be written as the sum of its integral part and fractional part. Let $x \in$ IR. Using Archimedean property of real numbers (It states that given a real number x there exists an integer n, n ≥ x) and well-ordering property of natural numbers (It states that every non-empty subset of natural numbers has the least element), there exists smallest integer n_x such that $x \le n_x$. This integer n_x is called the integral value of x and is denoted by [x]. Since $[x] \le x < [x] + 1$, $0 \le x - [x] < 1$. x – [x] is called the fractional part of x, and is denoted by r_x. By definition $S^1 = \{(\cos x, \sin x) : x \in [0, 2\pi)\}$. It can be seen that $S^1 = \{(\cos 2\pi y, \sin 2\pi y) : y \in [0, 1)\}$. Let $x \in$ IR. Then $x = [x] + r_x$ where $r_x \in [0, 1)$. Therefore $S^1 = \{(\cos 2\pi x, \sin 2\pi x) : x \in IR\}$. For $x \in$ IR, $e^{ix} = \cos x + i\sin x$ is a periodic function with periodicity 2π. There is (the covering map) π : IR → C = the set of complex numbers, defined as $\pi(x) = e^{2\pi ix} = \cos 2\pi x + i\sin 2\pi x$. As a point of IRxIR, $\pi(x) = (\cos 2\pi x, \sin 2\pi x)$. Thus π is also a function from IR to IRxIR. It can be seen (e.g. see [1]) that $\pi(x+n) = \pi(x)$ for every integer n. Thus π is a periodic function with periodicity 1. Since $S^1 = \{(\cos 2\pi x, \sin 2\pi x) : x \in IR\}$, the function π, maps IR onto the circle $S^1 \subset$ IRxIR, or C. For every integer n, $\pi : [n, n+1) \to S^1$ is one-one and onto. In particular, $\pi : [0, 1) \to S^1$ is one-one and onto. If we identify 0 and 1 of [0, 1], then S^1 can be identified with [0, 1]. A circle map is a continuous function $f : S^1 \to S^1$. For example, for a fixed ω, $0 < \omega < 2\pi$, if we define $f_\omega : S^1 \to S^1$ as $f_\omega(\cos x, \sin x) = (\cos(x+\omega), \sin(x+\omega))$, f_ω is a circle map. Also, for a fixed ω, $0 < \omega < 2\pi$, if we define $f_\omega^* : S^1 \to S^1$ as $f_\omega^*(\cos x, \sin x) = (\cos(x+2\pi\omega), \sin(x+2\pi\omega))$, f_ω^* is a circle map. If we define, for $x \in$ IR, F : IR → IR, $F(x) = x + \omega/2\pi$, then we note that that $\pi \circ F = f_\omega \circ \pi$. Let $x \in$ IR. $\pi(F(x)) = \pi(x + \omega/2\pi) = (\cos 2\pi(x + \omega/2\pi), \sin 2\pi(x + \omega/2\pi)) = (\cos(2\pi x+\omega), \sin(2\pi x+\omega))$. $\pi(x) = (\cos 2\pi x, \sin 2\pi x)$. $f_\omega(\pi(x)) = f_\omega(\cos 2\pi x, \sin 2\pi x) = (\cos(2\pi x+\omega), \sin(2\pi x+\omega))$. Such a function F is called a lift of the circle map. If we define F^* : IR → IR, $F^*(x) = x + \omega$, for $x \in$ IR, then it can be seen that $\pi \circ F^* = f_\omega^* \circ \pi$. As $F^*(x) = x + \omega$, $\pi(F^*(x)) = \pi(x+\omega) = (\cos 2\pi(x+\omega), \sin 2\pi(x+\omega))$. $\pi(x) = (\cos 2\pi x, \sin 2\pi x)$. $f^*(\pi(x)) = (\cos 2\pi(x+\omega), \sin 2\pi(x+\omega))$. Thus F^* is called a lift of the circle map f_ω^*. Thus the function π plays a significant role in the study of dynamics on circle maps. In the definition of lift of a circle map the function $F : IR \to IR$ is assumed to be continuous. Some results about the function F are obtained in [1] without assuming the continuity.

For each $x \in IR$, $x = [x] + r_x$ with $0 \leq r_x < 1$. This gives a function $r : IR \to [0, 1)$ taking x to r_x. The function r is called the fractional part function. The fractional part function apart from being used in proving some properties of subsets of IR (as seen below) is helpful in some part of the study of dynamics of circle maps, in particular, and functions in dynamical systems in general.

In this article we shall study some subsets of IR where the fractional part function is continuous.

The continuity of a real-valued function of a real variable is defined using ε–δ. Suppose the domain Y of a real-valued function of a real variable is a proper subset of IR. If Y is an interval, the same ε–δ definition of continuity works except at the end point(s), where we talk of left-hand/right-hand continuity depending upon the end point. When the domain Y is not necessarily an interval, we need an exact definition of continuity, to be precise in our considerations. An easy way for that is to consider subspace topology. But that definition is theoretical and not easy to apply. We have an equivalent definition of continuity of a real-valued function of a real variable defined on Y, in terms of ε–δ, which is easy to use.

2. DEFINITIONS AND NOTATION

IR is used to denote the real numbers. **IN** is the set of natural numbers. **Z** is the set of integers. A **circle** in the plane IRxIR is the boundary of the disc with centre at the origin of the plane and unit radius, it is denoted by $\mathbf{S^1}$. $S^1 = \{(\cos x, \sin x) : x \in [0, 2\pi)\} = \{(\cos 2\pi y, \sin 2\pi y) : y \in [0, 1)\} = \{(\cos 2\pi x, \sin 2\pi x) : x \in IR\}$. A **circle map** is a continuous function $f : S^1 \to S^1$. **Lift** of a circle map $f : S^1 \to S^1$ is a continuous function $F : IR \to IR$ such that (i) $\pi \circ F = f \circ \pi$ (ii) there exists some $k \in Z$ such that $F(x + 1)\ F(x) + k$ for every $x \in IR$.

Archimedean property of real numbers : Given a real number x there exists an integer n, $n \geq x$. **Well-ordering property of natural numbers** : Every non-empty subset of natural numbers has the least element. Let $x \in IR$. Using Archimedean property of real numbers and well-ordering property of natural numbers, there exists **smallest integer $\mathbf{n_x}$** such that $x \leq n_x$. This integer n_x is called the **integral value** of x. **[x]** is used to denote n_x. Since $[x] \leq x < [x] + 1$, $0 \leq x - [x] < 1$. $x - [x]$ is called the **fractional part** of x, and is denoted by $\mathbf{r_x}$. For each $x \in IR$, $x = [x] + r_x$ with $0 \leq r_x < 1$. This gives a function $\mathbf{r : IR \to [0, 1)}$ taking x to r_x. The function r is called the **fractional part function**.

Let $Y \subset IR$. **cl(Y)** denotes the closure of Y. A collection $\{Y_j : j \in J\}$ of subsets of IR is called **separated** or pair wise separated if, for every $j, k \in J$, $j \neq k$, $cl(Y_j) \cap Y_k = \phi$. Let $0 < s < 1$. Let $\mathbf{A_s} = \cup\{[n, n + s] : n \in Z\}$. Let $\mathbf{A_s^o} = \cup\{(n, n + s) : n \in Z\}$. Let $\mathbf{C_s} = \cup\{(n + s, n + 1) : n \in Z\}$. Let $\mathbf{C_s^*} = C_s \cup Z = \cup\{[n + s, n + 1) : n \in Z\}$. Let $\mathbf{B_s} = \{x \in IR : r(x) \leq s\}$. Let $\mathbf{B_s^*} = \{x \in IR : r(x) \geq s\}$. Let $\mathbf{E_s} = \{x \in IR : r(x) = s\} = B_s \cap B_s^*$. For $n \in Z$ and $0 < s < 1$, let $\mathbf{A_s^n} = [n, n + s]$ and $\mathbf{V_s^n} = [n, n + s)$. We can define A_s and A_s^n, also for $s = 0$; A_0 turns out to be Z and $A_0^n = \{n\}$. Let $Y \subset IR$. Let $g : Y \to IR$. Let $x \in Y$, $g : Y \to IR$ is continuous at x if $g : (Y, \tau^*) \to IR$ is continuous at x, where τ^* is the induced topology of the usual topology of IR. $g : Y \to IR$ is continuous if $g : Y \to IR$ is continuous at every $x \in Y$.

3. PRELIMINARIES

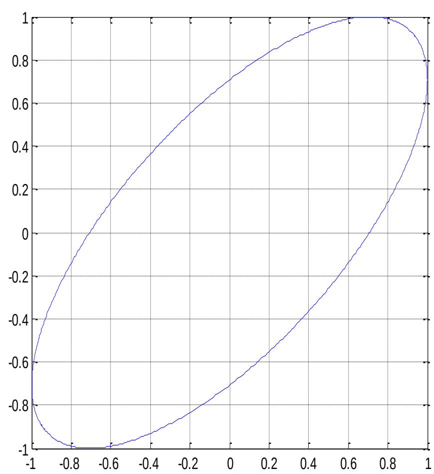
As seen in the introduction, for a fixed ω with $0 < \omega < 2\pi$, we have two circle maps f_ω and f_ω^*, defined as $f_\omega(\cos x, \sin x) = (\cos(x+\omega), \sin(x+\omega))$ and $f_\omega^*(\cos x, \sin x) = (\cos(x+2\pi\omega), \sin(x+2\pi\omega))$. $F : IR \to IR$, $F(x) = x + \omega/2\pi$, and $F^* : IR \to IR$, $F^*(x) = x + \omega$ are lifts of f_ω and f_ω^* respectively.

For $\omega = \pi/4$, $\pi/2$ and $3\pi/4$, we have the following graphs of f_ω. Since ω is taking a particular value, we can write f in place of f_ω.

$\omega = \pi/4$

```
x=0:0.01: pi;
x1=cos(x);
y1=cos(x+pi/4);
x2=sin(x);
y2=sin(x+pi/4);
plot(x1,y1)
grid on
```

$\omega = \pi/4$



$\omega = \pi/2$

x=0:0.01: pi;

x_1=cos(x);

y_1=cos(x+pi/2);

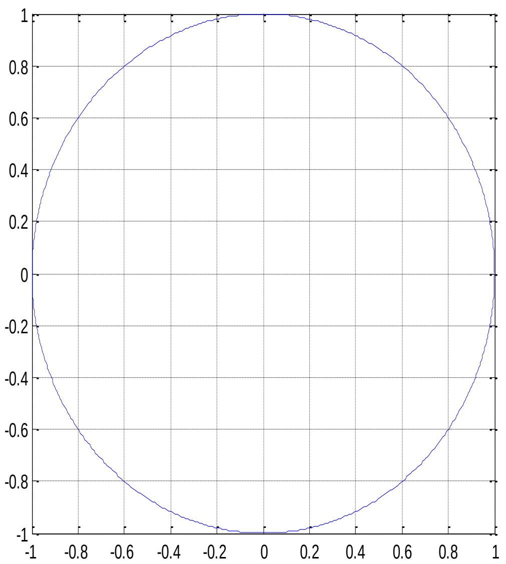
x_2=sin(x);

y_2=sin(x+pi/2);

plot(x_1,y_1)

grid on

$\omega = \pi/2$



$\omega = 3\pi/4$

x=0:0.01: pi;

x_1=cos(x);

y_1=cos(x+3*pi/4);

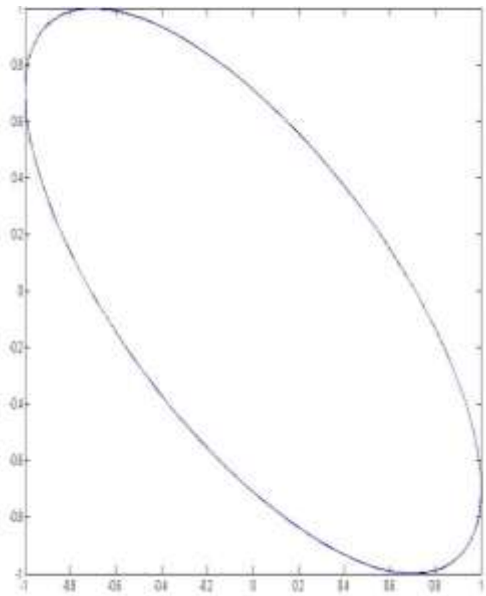
x_2=sin(x);

y_2=sin(x+3*pi/4);

plot(x_1,y_1)

grid on

$\omega = 3\pi/4$



Remark.3.1. IR = $\cup\{V_1^n : n \in Z\} = \cup\{[n, n + 1) : n \in Z\}$.

Proof. Let $x \in$ IR. $x = [x] + r(x)$ with $0 \le r(x) < 1$. Therefore $x \in [m, m + 1)$, where $m = [x]$. Thus IR = $\cup\{[n, n + 1) : n \in Z\}$.

Remark.3.2. (i) For every s, $0 \le s < 1$, $A_s = \cup\{A_s^n : n \in Z\}$. (ii) IR = $\cup\{[n, n + s) : n \in Z, 0 < s < 1\}$. (iii) IR = $\cup\{A_s^n : n \in Z, 0 < s < 1\}$.

Proof. (i) It follows by definitions of A_s and A_s^n. (ii) Let $x \in$ IR. $x = [x] + r(x)$ with $0 \le r(x) < 1$. Let t be such that $0 \le r(x) < t < 1$. Therefore $x \in [m, m + t)$, where $m = [x]$. Thus IR = $\cup\{[n, n + s) : n \in Z, 0 < s < 1\}$. (iii) By (ii), IR = $\cup\{[n, n + s) : n \in Z, 0 < s < 1\} \subset \cup\{A_s^n : n \in Z, 0 < s < 1\}$. So (iii) follows.

Lemma.3.3. For every s, $0 < s < 1$, $A_s = B_s$.

Proof. Let $x \in A_s$. Then $x \in [n, n + s]$ for some $n \in Z$. Since $n \le x \le n + s < n + 1$, $n = [x]$. Since $x = [x] + r(x)$ and $n + r(x) \le n + s$, therefore $r(x) \le s$. So $x \in B_s$. Conversely, let $x \in B_s$. $[x] \le x = [x] + r(x) \le [x] + s$ as $r(x) \le s$. Therefore $x \in [[x], [x] + s]$. So $x \in A_s$.

Remark.3.4. For every s, $0 < s < 1$, $E_s = \{n + s : n \in Z\}$.

Proof. Let $x \in E_s$. Since $r(x) = s$, $x = [x] + s$. Therefore $x \in \{n + s : n \in Z\}$. Conversely, let $x \in \{n + s : n \in Z\}$. Then $x = m + s$ for some $m \in Z$. So $x - m = s$. Since $[x - m] = - m + [x]$ and $0 < s < 1$, therefore $m = [x]$. This implies that $r(x) = s$. Therefore $x \in E_s$.

Lemma.3.5. Let $H \subset$ IR. Let $h : H \to$ IR. Let $x \in H$. If there exists a $\delta^* > 0$ such that for $y \in (x - \delta^*, x +$

δ^*), (i) $x - y = h(x) - h(y)$, or (ii) $|h(x) - h(y)| \leq \lambda|x - y|$ for some fixed $\lambda > 0$, then h is continuous at x.

Proof. (i) Let $\varepsilon > 0$. Let $\delta = \min\{\varepsilon, \delta^*\}$. Let $|y - x| < \delta$. Then $y \in (x - \delta^*, x + \delta^*)$ as $\delta \leq \delta^*$. Therefore by the given condition (i), $|h(x) - h(y)| = |x - y| < \delta \leq \varepsilon$. (ii) Let $\varepsilon > 0$. Let $\delta = \min\{\varepsilon/\lambda, \delta^*\}$. Let $|y - x| < \delta$. Then $y \in (x - \delta^*, x + \delta^*)$ as $\delta \leq \delta^*$. Therefore by the given condition (ii), $|h(x) - h(y)| \leq \lambda|x - y| < \delta\lambda \leq \varepsilon$.

4. RESULTS

Lemma.4.1. Let $\{Y_j : j \in J\}$ be a collection of subsets of IR such that $\cup\{Y_j : j \in J\}$ is closed. Then $\{Y_j : j \in J\}$ are pair wise separated iff $\{Y_j : j \in J\}$ are pair wise disjoint and each Y_j is closed.

Proof. If $\{Y_j : j \in J\}$ are pair wise disjoint and each Y_j is closed then clearly $\{Y_j : j \in J\}$ are pair wise separated. Now suppose that $\{Y_j : j \in J\}$ are pair wise separated. Let $k \in J$. Let $x \in cl(Y_k)$. Suppose $x \notin Y_k$. For $j \in J$, $j \neq k$, since $cl(Y_k) \cap Y_j = \phi$, $x \notin Y_j$. Thus $x \notin \cup\{Y_j : j \in J\}$. Therefore $x \in B = IR - \cup\{Y_j : j \in J\}$. Since B is open and $x \in cl(Y_k)$, $B \cap Y_k \neq \phi$. But $B \cap Y_k = \phi$ as $B \subset IR - Y_k$.

Corollary.4.2. Let $\{B_j : j \in J\}$ and $\{Y_j : j \in J\}$ be such that $IR - \cup\{Y_j : j \in J\} = \cup\{B_j : j \in J\}$. If $\cup\{B_j : j \in J\}$ is open then $\{Y_j : j \in J\}$ are pair wise disjoint and each Y_j is closed iff $\{Y_j : j \in J\}$ are pair wise separated.

Proof. By the given condition $\cup\{Y_j : j \in J\}$ is closed. Therefore the result follows by Lemma.4.1.

Lemma.4.3. Let $IR = \cup\{Y_j : j \in J\}$ be such that $Y_j \cap Y_k = \phi$ for every j, $k \in J$, $j \neq k$. If for each $j \in J$, $Y_j = H_j \cup K_j$ with $H_j \cap K_j = \phi$, then $IR - \cup\{H_j : j \in J\} = \cup\{K_j : j \in J\}$.

Proof. Let j, $k \in J$, $k \neq j$. Since $Y_j \cap Y_k = \phi$, $Y_j \cap H_k = \phi$. Thus $Y_j \subset IR - H_k$. So $Y_j - H_k = Y_j \cap (IR - H_k) = Y_j$. This implies that $\cap\{Y_j - H_k : k \in J, k \neq j\} = Y_j$. Therefore, $\cap\{Y_j - H_k : k \in J\} = Y_j - H_j$. Now $IR - \cup\{H_j : j \in J\} = (\cup\{Y_j : j \in J\}) - (\cup\{H_k : k \in J\}) = \cup\{(Y_j - (\cup\{H_k : k \in J\}) : j \in J\} = \cup\{\cap(Y_j - H_k : k \in J) : j \in J\} = \cup\{Y_j - H_j : j \in J\} = \cup\{K_j : j \in J\}$, as $Y_j - H_j = K_j$.

Lemma.4.4. For every s, $0 < s < 1$, $IR - A_s = C_s = \cup\{(n + s, n + 1) ; n \in Z\}$.

Proof. By Remark.3.1, $IR = \cup\{[n, n + 1) : n \in Z\}$. Let J = Z. Let, for $n \in Z$, $Y_n = [n, n + 1)$. Then $Y_n \cap Y_k = \phi$ for every n, $k \in Z$, $n \neq k$. Let $H_n = [n, n + s]$ and $K_n = (n + s, n + 1)$. Then $Y_n = H_n \cup K_n$ and $H_n \cap K_n = \phi$. Therefore by Lemma.4.3, $IR - A_s = C_s = \cup\{(n + s, n + 1) : n \in Z\}$.

Remark.4.5. For every s, $0 < s < 1$, $B_s^* = \cup\{[n + s, n + 1) : n \in Z\}$.

Proof. $B_s^* = \{x : r(x) > s\} \cup \{x : r(x) = s\}$. Therefore, by definition of B_s, $B_s^* = (IR - B_s) \cup E_s = (IR - A_s) \cup E_s$ as $B_s = A_s$ by Lemma.3.3. Since, by Lemma.4.4, $IR - A_s = C_s = \cup\{(n + s, n + 1) ; n \in Z\}$ and $E_s = \{n + s : n \in Z\}$, $B_s^* = \cup\{[n + s, n + 1) ; n \in Z\}$.

Remark.4.6. Let $0 < s < 1$. For every $n \in Z$, n is a limit point of B_s^*, so B_s^* is not closed.

Proof. By Remark.4.5, $B_s^* = \cup\{[n + s, n + 1) : n \in Z\}$. Since $n \in Z$ iff $n - 1 \in Z$, $B_s^* = \cup\{[n - 1 + s, n) : n \in Z\}$. For $n \in Z$, n is a limit point of B_s^*, so B_s^* is not closed.

Remark.4.7. $IR - A_s^o = C_s^* = \cup\{[n + s, n + 1) ; n \in Z\}$, where $A_s^o = \cup\{(n, n + s) : n \in Z\}$.

Proof. $IR = \cup\{[n, n + 1) : n \in Z\}$. Let J = Z. Let for $n \in Z$, $Y_n = [n, n + 1)$. Then $Y_n \cap Y_k = \phi$ for every n, $k \in Z$, $n \neq k$. Let $H_n = (n, n + s)$ and $K_n = [n + s, n + 1)$. Then $Y_n = H_n \cup K_n$ and $H_n \cap K_n = \phi$. Therefore by Lemma.4.3, $IR - A_s^o = C_s^* = \cup\{[n + s, n + 1) ; n \in Z\}$.

Remark.4.8. The Lemma.4.4 can also be proved using the function r i.e using Lemma.3.3. We give the proof below.

Lemma.4.9. (i) $IR - A_s = C_s = \cup\{(n + s, n + 1) ; n \in Z\}$. (ii) A_s is closed.

Proof. (i) Since by Lemma.3.3, $A_s = B_s$, we prove that $IR - B_s = \cup\{(n + s, n + 1) : n \in Z\}$. $IR - B_s = \{x \in IR ; r(x) > s\}$. Let $x \in IR - B$. So $r(x) > s$. $x = [x] + r(x)$. We claim $x \in ([x] + s, [x] +1)$. Since $r(x) > s$, $[x] + r(x) > [x] + s$. Since $r(x) < 1$, $[x] + r(x) < [x] + 1$. Therefore $[x] + s < x < [x] + 1$. Hence $x \in ([x] + s, [x] + 1)$. For the converse, suppose that $x \in (n + s, n + 1)$ for some $n \in Z$. Since $n + s < x < n + 1$, $n = [x]$. So $r(x) = x - [x] > s$. Therefore, $x \in IR - B_s$. (ii) Since $\cup\{(n + s, n + 1) ; n \in Z\}$ is open, by (i), $IR - A_s$ is open. Therefore, A_s is closed.

The following is the direct proof that A_s is closed. We need the following Remark for that.

Remark.4.10. Let a, x, $b \in IR$ such that $a < x < b$. Let $\delta = \min\{x - a, b - x\}$ then $a \leq x - \delta$ and $x + \delta \leq b$. So $(x - \delta, x + \delta) \subset (a, b)$.

Proof. Since $\delta = \min\{x - a, b - x\}$, $\delta \leq x - a$ and $\delta \leq b - x$. Therefore $a \leq x - \delta$ and $x + \delta \leq b$. Therefore $(x - \delta, x + \delta) \subset (a, b)$.

Proof of A_s is closed: Suppose $x \notin A_s$. So $x \notin [[x], [x] + s]$. Therefore $r(x) > s$ as $x = [x] + r(x)$. Thus $[x] + s < x < [x] + 1$. Let $\delta = \min\{r(x) - s, 1 - r(x)\}$. Since $x - ([x] + s) = r(x) - s$, and $[x] + 1 - x = 1 - r(x)$, by Remark.4.10, $(x - \delta, x + \delta) \subset ([x] + s, [x] + 1)$. Therefore $(x - \delta, x + \delta) \cap [[x], [x] + s] = \phi$. Let $n \in Z$, $n \neq [x]$. Either $n < [x]$ or $[x] + 1 \leq n$. Let $n < [x]$. By Remark.4.10, $s \leq r(x) - \delta$. Therefore, $n + s < x - \delta$. Suppose $[x] + 1 \leq n$. Using Remark.4.10, $x + \delta \leq [x] +1 \leq n$. Thus $(x - \delta, x + \delta) \cap [n, n + s] = \phi$ for every n

$\in$ Z. Therefore $(x - \delta, x + \delta) \cap A_s = \phi$. So x is not a limit point of A_s. Hence A_s is closed.

As mentioned in the introduction, we need to have a working definition of the continuity of a real-valued function of a real variable is defined using ε–δ, when the domain Y of the function is a proper subset of IR. If Y is an interval, the same ε–δ definition of continuity works except at the end point(s). When the domain Y is not necessarily an interval, we have the following definition which works even when Y is an interval. But first we have the theoretical definition. Then we have an equivalent definition of continuity of a real-valued function of a real variable in terms of ε–δ, which we use later.

Remark.4.11. Let $Y \subset$ IR. g : Y $\to$ IR. Let $x \in Y$. (i) g is continuous at x iff, for given $\varepsilon > 0$, there exists $\delta > 0$ such that for $y \in (x - \delta, x + \delta) \cap Y$, $|g(x) - g(y)| < \varepsilon$. (ii) If there exists $\delta > 0$ such that for $y \in (x - \delta, x + \delta) \cap Y$, $|g(x) - g(y)| \le |x - y|$, then g is continuous at x.

Proof. (i) It follows as $(x - \delta, x + \delta) \cap Y$ is open in the induced topology on Y. (ii) For given $\varepsilon > 0$, if we take $\delta^* = \min\{\varepsilon, \delta\}$, then g is continuous at x.

Remark.4.12. Let $H \subset Y \subset$ IR. Let g : Y $\to$ IR. Let $x \in$ H. (i) If g : H $\to$ IR is not continuous at x, then g : Y $\to$ IR is not continuous at x. (ii) The converse of (i) is not true. That is, if g : Y $\to$ IR is not continuous at x, then g : H $\to$ IR may be continuous at x.

Proof. (i) g : H $\to$ IR is not continuous at x, so by Remark.4.11, there exists some $\varepsilon > 0$ such that whatever $\delta > 0$ we take there exists $y \in (x - \delta, x + \delta) \cap H$ such that $|g(x) - g(y)| \ge \varepsilon$. Since $H \subset Y$, $y \in (x - \delta, x + \delta) \cap Y$. Therefore, in view of Remark.4.11, g : Y $\to$ IR is not at x. (ii) Take H = Z and Y = IR. Every g : Z $\to$ IR is continuous. But every g : IR $\to$ IR is not continuous.

Remark.4.13. Let x, y $\in$ IR. If [x] = [y], then x – y = r(x) – r(y). r(x) < r(y) iff x < y.

Proof. x = [x] + r(x) and y = [y] + r(y). Therefore x – y = r(x) – r(y) as [x] = [y]. Now it follows that r(x) < r(y) iff x < y.

Proposition.4.14. Let $A_s = \cup\{[n, n + s] ; n \in Z\}$. r : A_s $\to$ IR is continuous.

Proof. Let $x \in A_s$. First we find $\delta > 0$ such that for $y \in (x - \delta, x + \delta) \cap A_s$, [y] = [x]. $x \in [m, m + s]$ for some $m \in$ Z. Case(i) $x \in (m, m + s)$. Let $\delta = \min\{x - m, m + s - x\}$. By Remark.4.10, $(x - \delta, x + \delta) \subset (m, m + s)$. Let $y \in (x - \delta, x + \delta) \cap A_s$. Then $y \in (m, m + s)$. Therefore [y] = [x] as s < 1. Case(ii) x = m + s. Take $\delta = ½(\min\{s, (1-s)\})$. By Remark.4.10, $(x - \delta, x + \delta) \subset (m, m+1)$. Let $y \in (x - \delta, x + \delta) \cap A_s$. Then $y \in (m, m+1)$. Therefore [y] = m = [x] as 0 < s < 1. Case(iii) x = m. Take $\delta = ½(\min\{s, (1 - s)\})$. Now by Remark.4.10, $(x - \delta, x + \delta) \subset (m - 1, m + s)$. Let $y \in (x - \delta, x + \delta) \cap A_s$. Then $y \in [m, m + s)$ as $(m - 1, m + s) \cap A_s \subset [m, m + s)$. Therefore [y] = [x]. Thus in every case, for $y \in (x - \delta, x + \delta) \cap A_s$, [y] = [x]. By Remark.4.13, x – y = r(x) – r(y). Now, by Remark.4.11 (ii), r is continuous at x.

Remark.4.15. By Proposition.4.14, for every s with 0 < s < 1, r is continuous on A_s. The collection $\{A_s : s \in IR, 0 < s < 1\}$ is a totally ordered subset of the p.o. set (IR, $\subset$) because for s, t $\in$ IR, 0 < s, t < 1, $A_s \subset A_t$, or $A_t \subset A_s$ depending upon $s \le t$, or $t \le s$. Let $x \in$ IR. x = [x] + r(x). Since $0 \le r(x) < 1$, $x \in A_t$ for every t such that r(x) < t. Therefore $\cup\{A_s : s \in IR, 0 < s < 1\}$ = IR. By definition $A_s^o = \cup\{(n, n + s) : n \in Z\}$. It can be seen that $\cup\{A_s^o : s \in IR, 0 < s < 1\}$ = IR – Z.

Remark.4.16. We have seen above that, for s = 0, A_s = Z, i.e. A_0 = Z. It can be seen (below) that r is continuous also on A_0.

Remark.4.17. (i) r is continuous on IR – Z. (ii) r is continuous on Z. (iii) r is continuous on H, if $H \subset$ IR – Z or Z.

Proof. (i) Let $x \in$ IR – Z. Since [x] < x < [x] +1, by Remark.4.10 there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset ([x], [x] + 1)$. Therefore, for $y \in (x - \delta, x + \delta)$, [y] = [x]. By Remark.4.13, x – y = r(x) – r(y). Therefore r is continuous at x. Therefore r is continuous on IR – Z. (ii) Let $m \in$ Z. Let $\varepsilon > 0$. For $\delta < 1$, $(m - \delta, m + \delta) \cap Z = \{m\}$, therefore r(y) – r(m) = 0 for every $y \in (m - \delta, m + \delta) \cap Z$. (iii) Restriction of a continuous function is continuous.

5. CONCLUSION

In the above considerations there are many subsets of IR on which the fractional part function r is continuous. There may be other subsets of IR on which r is continuous. But we do not have any more information about such subsets of IR. It is worth investigating to know other subsets of IR on which r is continuous, or to have some information about such subsets.

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