A Study of New Results for the Existence in Vector Spaces of Solutions of Generalized Variations Such As Inequalities for Multi-Functions

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Abstract – This article examines a class of widespread non-linear vector inequalities and nearly variable characters with maps in Hausdorff's topological vector space areas which include popular, non-linear, mixed variable inequalities, widespread, mixed, near-variable inequalities, etc. We obtain solutions for generalized, nonlinear non-linear vector inequalities in local, convex, topological vector spaces, using a fixed-point theorem.

Key Words – Generalized Nonlinear Vector, Variational, Inequality, Maximal Element Theorem, Upper Semi Continuous, Diagonal Convexity, Locally Convex, Topological Vector Space

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INTRODUCTION

In a finite-dimensional euclidean space Giannessi first introduced a vector variational inequality (VVI). Giannessi[6]. This is a generalisation by multi-criters of the scalary variational inequality in the vector case. The fundamental existence theorem of solutions for nonlinear variations was studied by Browder in 1966[4] first and demonstrated. Recently, the outcome of the Browder 's work, Liu et al . [8], Ahmad and Irfan[1], Husain and Gupta [7] and Xiao et al. [16], Zhao et al.[18] expanded to broader nonlinear variance inequality. In this document we regard a generalised nonlinear vector as virtually variational inequality and we detect some findings of existence in the locally convex theorem of the topological vector spaces. Let E be a locally convex topological vector space and K be a nonempty convex subset of a Hausdorff topological vector space F. Let Y be a subset of continuous function space L(F, E) from F into E, where L(F, E) is equipped with a σ-topology. Let int A and coA denotes the interior and convex hull of a set A, respectively. Let $C : K \to 2^E$ be a set-valued mapping such that intC(x) , ∅ for each x ∈ K, $\eta : K \times K \to F$ be a vector-valued mapping.

Let $Z : L(F, E) \times L(F, E) \times L^{(F, E)} \to 2^{L(F,E)}$, $H : K \times K \to 2^E$, $D : K \to 2^K$ and M,R, $T : K \to 2^Y$ be set-valued mappings. We now consider the following class of generalized nonlinear vector quasi-variational-like inequality problems (GNVQVLIP, in short), which is to find x ∈ K such that x ∈ D(x) and ∀y ∈ D(x), ∃ u ∈ M(x), v ∈ R(x), w ∈ T(x) satisfying

$$\langle Z(u, v, w), \eta(y, x)\rangle + H(x, y) \not\subseteq -\text{int}C(x),$$

Where ⟨u, x⟩ denotes the evaluation of u ∈ L(F, E) at x ∈ F. By the corollary of the Schaefer [12], L(F, E) becomes a locally convex topological vector space. By Ding and Tarafdar [5], the bilinear map $\langle \cdot, \cdot \rangle: L(K, E) \times K \to E$ is continuous.

Generalized nonlinear vector quasi-variational-like inequality includes many important variational inequalities as special cases:

If D(x) = K, GNVQVLIP reduces to the problem of finding x ∈ K such that u ∈ M(x), v ∈ R(x), w ∈ T(x) satisfying

$$\langle Z(u,v,w), \eta(y,x)\rangle + H(x,y) \not\subseteq -\text{int}C(x), \quad \forall y \in K.$$

If Z, H are two single-valued mappings and D(x) = K where K is a nonempty convex subset of a Banach space $F, E = \mathbb{R}, F^* = L(F,E)$. Let $B, A : K \to F^*$ and $P, Q : F^* \times F^* \to F^*$ be four mappings, b: K × K → R be a real-valued functional, R, T : K → F ∗ be two single-valued mappings. For a given ω∗ ∈ F∗ , if M(x) = P(B(x), A(x)), Z(u, v, w) = Q(T(x),R(x)) − M(x) + ω∗ , H(x, y) = b(x, y) − b(x, x), C(x) = R+ for all x ∈ K, then GNVQVLIP reduces to the problem (see e.g. [18]) of finding x ∈ K such that

$$\langle Q(T(x), R(x)) - M(x) + \omega^*, \ \eta(y,x)\rangle + b(x,y) - b(x,x) \geq 0, \quad \forall y \in K.$$

If Z, H are two single-valued mappings and D(x) = K where K is a nonempty convex subset of a real Hilbert space F, E = R, C(x) = R+ for all x ∈ K. If H(x, y) = ϕ(y, x) + ϕ(x, x) and T(x) = ∅ for all x ∈ K, Then GNVQVLIP reduces to the problem (see e.g. [10]) of finding x ∈ K such that ∃ u ∈ M(x) and v ∈ R(x) satisfying

$$\langle Z(u,v), \eta(y,x)\rangle + \phi(y,x) + \phi(x,x) \geq 0, \quad \forall y \in K.$$

For suitable and appropriate choice of mappings M,R, T,Z, H, η, one can obtain various new and previously known variational inequality problems. The technical instrument in our proof is similar to that employed by, Tian [14] and Peng and Yang [11].

2. PRELIMINARIES

Definition 2.1. ([15]) Let *A* and *B* be two topological vector spaces and T : A → 2^B be a set-valued mapping. Then,

(i) T is said to have open lower sections if the set

T − (y) = {x ∈ A : y ∈ T(x)} is open in A f or every y ∈ B;

(ii) T is said to be lower semi continuous if for each x ∈ A and each open set C in B with T(x) ∩ C , ∅, there exists an open neighborhood O of x in A such that

T(u) ∩ C , for each u ∈ O;

(iii) T is said to be upper semi continuous if for each x ∈ A and each open set C in B with T(x) ⊂ C, there exists an open neighborhood O of x in A such that

T(u) ⊂ C f or each u ∈ O;

(iv) T is said to be continuous if it is both lower and upper semi continuous;

(v) T is said to be closed if any net {xα} in A such that xα → x and any net {yα} in B such that yα → y and yα ∈ T(xα) for any α , we have y ∈ T(x).

Definition 2.2. ([9]) Let F and E be two topological vector spaces, K a convex subset of topological vector space F. A (set-valued) mapping θ: K × K → (2^E) E is called (generalized) vector 0-diagonally convex if for any finite subset $\Lambda = \{x_1, x_2 \ldots, x_n\}$ of *K* and any

$$x = \sum_{i=1}^{n} \alpha_i x_i \text{ with } \alpha_i \geq 0 \text{ for } i = 1, \ldots, n \text{ and } \sum_{i=1}^{n} \alpha_i = 1,$$

$$\sum_{i=1}^{n} \alpha_i \theta(x, x_i)(\not\subseteq) \notin -\text{int}C(x).$$

Lemma 2.3. ([3]) Let A and B be two topological spaces. If T : A → 2^B is an upper semi continuous set-valued mapping with closed values, then T is closed.

Lemma 2.4. ([13]) Let A and B be two topological spaces and T : A → 2^B is an upper semi continuous set-valued mapping with compact values. Suppose {xα} is a net in A such that xα → x0. If yα ∈ T(xα) for each α , then there are a y0 ∈ T(x0) and a subnet {yβ} of {yα} such that yβ → y0.

Lemma 2.5. ([17]) Let A and B be two topological spaces. Suppose T : A → 2^B and K : A → 2^B are set-valued mappings having open lower sections, then

(i) a set-valued mapping J : A → 2^B defined by , for each x ∈ A, J(x) = coT(x) has open lower sections;

(ii) a set-valued mapping θ : A → 2^B defined by , for each x ∈ A, θ(x) = T(x) ∩ K(x) has open lower sections.

Let I be an index set, Fi a Hausdorff topological vector spaces for each i ∈ I. Let {Ki} be a family of nonempty compact convex subsets with each Ki in Fi . Let K = Πi∈I Ki and F = Πi∈I Fi. The following fixed-point theorem is needed in this paper.

Lemma 2.6. ([2]) For each i ∈ I, let Ti : K → 2^{Ki} be a set-valued mapping. Assume that the following conditions hold.

(i) For each i ∈ I, Ti is convex set-valued mapping;

(ii) $K = \bigcup\{intT_i^-(x_i) : x_i \in K_i\}.$

Then there exist x¯ ∈ K such that x¯ ∈ T(x¯) = Πi∈I Ti(x¯), that is, x¯i ∈ Ti(x¯) for each i ∈ I, where x¯i is the projection of x onto K¯ i .

3. EXISTENCE OF SOLUTIONS

First, we prove the following existence theorem for GNVQVLIP.

Theorem 3.1. Let E be a locally convex topological vector space, K a nonempty compact convex subset of Hausdorff topological vector spaces F, Y a nonempty compact convex subset of L(F, E), which is equipped with a σ-topology. Let M, R, T : K → 2 Y be three upper semi continuous set-valued mappings with nonempty compact values. Assume that the following conditions are satisfied.

(i) D: $K \to 2^K$ is a nonempty convex set-valued mapping and has open lower sections;

(ii) for all $y \in K$, the mapping $\langle Z(., ., .), \eta(y, .)\rangle + H(., y) : L(F, E) \times L(F, E) \times L(F, E) \times K \times K \to 2^E$ is an upper semi continuous set-valued mapping with compact values;

(iii) $C : K \to 2^E$ is a convex set-valued mapping with intC(x) , ∅ for all $x \in K$;

(iv) $\eta : K \times K \to F$ is affine in the first argument and for all $x \in K$, $\eta(x, x) = 0$;

(v) $H : K \times K \to 2^E$ is a generalized vector 0-diagonally convex set-valued mapping;

(vi) Let $\Lambda(x) = \{y \in K : \langle Z(u, v, w), \eta(y, x)\rangle + H(x, y) \subseteq -\mathrm{int}C(x), \forall u \in M(x), v \in R(x), w \in T(x)\}$, $\forall x \in K$, the set {x ∈ K : coΛ(x) ∩ D(x) , ∅} is closed in K.

Then there exists a point x¯ ∈ K such that x¯ ∈ D(x¯), ∀y ∈ D(x¯), ∃ u ∈ M(x¯), v ∈ R(x¯), w ∈ T(x¯) :

$$\langle Z(u,v,w), \eta(y,\bar{x})\rangle + H(\bar{x},y) \not\subseteq -\mathrm{int}C(\bar{x}).$$

Proof. Define a set-valued mapping $Q : K \to 2^K$ by

$$Q(x) = \{y \in K : \langle Z(u,v,w), \eta(y,x)\rangle + H(x,y) \subseteq -\mathrm{int}C(x), \forall u \in M(x), v \in R(x), w \in T(x)\}, \quad \forall x \in K.$$

We first prove that x < coQ(x) for all x ∈ K. To see this, suppose, by way of contradiction, that there exists some point x¯ ∈ K such that x¯ ∈ coQ(x¯). Then there exists a finite subset {y1, y2, ..., yn} ⊂ Q(x¯) for x¯ ∈ co{y1, y2, ..., yn} such that

$$\langle Z(u,v,w), \eta(y_i,\bar{x})\rangle + H(\bar{x},y_i) \subseteq -\mathrm{int}C(\bar{x}), \quad i = 1,2....,n.$$

Since $\mathrm{int}C(\bar{x})$ is convex and η is affine in the first argument, for $i = 1,2,....,n$, $\alpha_i \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, $\bar{x} = \sum_{i=1}^{n} \alpha_i y_i$, we have

$$\left\langle Z(u,v,w), \eta\left(\sum_{i=1}^{n} \alpha_i y_i, \bar{x}\right)\right\rangle + \sum_{i=1}^{n} \alpha_i H(\bar{x}, y_i) \subseteq -\mathrm{int}C(\bar{x}).$$

Since $\eta(\bar{x},\bar{x}) = 0$ for all $x \in K$, we have

$$\sum_{i=1}^{n} \alpha_i H(\bar{x}, y_i) \subseteq -\mathrm{int}C(\bar{x}),$$

which contradicts the condition (v), so that x < coQ(x) for all x ∈ K.

We now prove that for each y ∈ K,

$$Q^{-}(y) = \{x \in K : \langle Z(u,v,w), \eta(y,x)\rangle + H(x,y) \subseteq -\mathrm{int}C(x), \ \forall u \in M(x), v \in R(x), w \in T(x)\},$$

We only need to prove that J(y) is closed for all y ∈ K. Let {xα} be a net in J(y) such that xα → x ∗ . Then there exist uα ∈ M(xα), vα ∈ R(xα) and wα ∈ T(xα) such that

$$\langle Z(u_\alpha, v_\alpha, w_\alpha), \eta(y, x_\alpha)\rangle + H(x_\alpha, y) \not\subseteq -\text{int}C(x_\alpha).$$

Since $M, R, T : K \to 2^Y$ are three upper semi continuous set-valued mappings with compact values, by Lemma 2.4, $\{u_\alpha\}, \{v_\alpha\}, \{w_\alpha\}$ have convergent subnets with limits, say u^*, v^*, w^* and $u^* \in M(x^*), v^* \in R(x^*)$ and $w^* \in T(x^*)$. Without loss of generality we may assume that $u_\alpha \to u^*, v_\alpha \to v^*, w_\alpha \to w^*$. Suppose that

$$z_\alpha \in \{\langle Z(u_\alpha, v_\alpha, w_\alpha), \eta(y, x_\alpha)\rangle + H(x_\alpha, y) \not\subseteq -\text{int}C(x_\alpha)\}.$$

Since $\langle Z(.,.,.), \eta(y,.)\rangle + H(.,y)$ is an upper semi continuous with compact values, by Lemma 2.4, there exists a $z^* \in \langle Z(u^*, v^*, w^*), \eta(y, x^*)\rangle + H(x^*, y)$ and a subnet $\{z_\beta\}$ of $\{z_\alpha\}$ such that $z_\beta \to z^*$. Hence,

$$J(y) = \{x \in K : \exists u \in M(x), v \in R(x), w \in T(x) \text{ such that } \langle Z(u,v,w), \eta(y,x)\rangle + H(x,y) \not\subseteq -\text{int}C(x)\},$$

is closed in K. So that Q− (y) is open for each y ∈ K. Therefore Q has open lower sections.

Consider a set-valued mapping $G : K \to 2^K$ defined by

$$G(x) = coQ(x) \cap D(x), \quad \forall x \in K.$$

Since D has open lower sections by hypothesis (i), we may apply Lemma 2.5 to assert that the set-valued mapping G also has open lower sections. Let

$$N = \{x \in K : G(x) \neq \emptyset\} \subset K.$$

There are two cases to consider. In the case N = ∅, we have

$$coQ(x) \cap D(x) = \emptyset, \quad \forall\, x \in K.$$

This implies that, ∀ x ∈ K,

$$Q(x) \cap D(x) = \emptyset.$$

On the other hand, by condition (i), and the fact K is a compact subset of F, we can apply Lemma 2.6, in the case that I = {1}, to assert the existence of fixed point x ∗ ∈ D(x ∗), we have

$$Q(x^*) \cap D(x^*) = \emptyset.$$

This implies that $\forall y \in D(x^*),\ y \notin Q(x^*)$. Hence, in this particular case, the assertion of the theorem holds. We now consider the case N, ∅. Define a set-valued mapping $S : K \to 2^K$ by

$$S(x) = \begin{cases} G(x), & x \in N, \\ D(x), & x \in K \setminus N. \end{cases}$$

Then, it is clear that ∀x ∈ K, S(x) is convex set-valued mapping and for each v ∈ K,

$$S^{-}(v) = G^{-}(v) \cup ((K \setminus N) \cap (D^{-}(v))).$$

Since D− (v), coQ− (v) are open in K and K \ N is open in K by condition (vi), we have S − (v) is open in K. This implies that S has open lower sections and satisfies all the conditions of Lemma 2.6. Therefore, there exists x ∗ ∈ K such that x ∗ ∈ S(x ∗). Suppose that x ∗ ∈ N, then

$$x^* \in coQ(x^*) \cap D(x^*),$$

so that $x^* \in coQ(x^*)$. This is a contradiction. Hence, $x^* \notin N$. Therefore,

$$x^* \in D(x^*), \text{ and } G(x^*) = \emptyset.$$

Thus

$$x^* \in D(x^*), \text{ and } coQ(x^*) \cap D(x^*) = \emptyset.$$

This implies

$$Q(x^*) \cap D(x^*) = \emptyset.$$

Consequently, the assertion of the theorem holds in this case.

CONCLUSION

Variational inequality theory has been a popular and efficient method for analysing and investigating a wide variety of issues, including elasticity, optimisation, markets, transport and structural studies, and the references. The present article investigates the nature of its solutions in a locally convex topological vector space in a new class of generalised nonlinear variationarities with fixed valued mappings.

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