Journal of Advances and Scholarly Researches in Allied Education
Vol. XIV, Issue No. 3, (Special Issue) February-2018, ISSN 2230-7540

A Study of Linear Algebra and Matrices Alternative Formulations in Vector Spaces

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Abstract* – *Linear algebra and its dimension are clearly defined and the quantity of separate spatial directions approximately defines vector spaces. Naturally, in mathematical analysis, vector spaces that have functions exist as function spaces. Generally, these vector spaces have several more structures, including a topology which enables exploration of closeness and continuity issues. These topologies are more widely used described by a standard or an internal product (with a notion of distance between two vectors). In the field of mathematical analyzes, this is particularly true of Banach and Hilbert Spaces. There is increasing use of vector spaces in math, science and engineering. They are the best linear-algebraic definition in linear equation systems. They provide the basis for Fourier expansion, used in image compression, and create an environment for partial differential equations for solution technology.

Key Words* – *Linear Algebra, Vector Spaces, Mathematics, Application, Problems, Linear Equations

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INTRODUCTION

An array of items called vectors, also known as linear space, may be added and multiplied by numbers (the "scalar") called scalars, together. Scalar numbers are mainly called true numbers, but vector spaces are usually present with a scalar multiplication by complex numbers, rational numbers or a field. Vector enhancement and scalar multiplication operations must meet certain so-called vector axioms. The term "Real Vector Space" is also used to determine the real or difficult number of the scalars.

Certain sets of Euclidean vectors are common examples of vector space. These are physical numbers, like forces, where a third of the two forces (of the same type) may be added and a second force vector is the multiplication of an actual multiplier by a force vector. Flat or three-dimensional space shifting vectors often form vector spaces in a common vein (but more geometrically). In spatial vectors, arrow-like objects do not necessarily appear in the examples above: vectors are known as abstract mathematical objects with unique properties and in some cases can be interpreted as arrows.

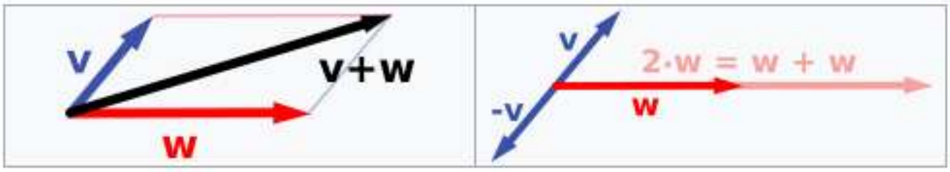
Linear algebra is one of the key branches of mathematics. In many practical problems, measured quantities are connected by linear equations to measurable quantities and can therefore be assessed with the traditional row operating methods. The methods to be used are often the most apparent (and most important). The theory's vector space is further refinement and results in the resolution of the linear difference equations, with the usual approach of non-linear equations using a repeated solution of the required linear equation, in order to obtain a convergent sequence of approximate solutions. Vector space is a set of vectors V that has two options and is known as vector objects as addition and scalar propagation. In addition, the theory of the vector spaces originally produced for the resolution of linear equations is widely used and based in many other branches of mathematics.

Historically, the first ideas leading to vectors can be traced back to the 17th century's analysis geometry, matrices, linear equation systems and euclidean vectors. In 1888 Giuseppe Peano suggested a new, more abstract study with a wider range of objects than Euclidean Space, yet the majority of the theories can be regarded as extending classical geometrical ideas like lines, planes and analogs of greater dimensions. The concept of vector space will first be explained by describing two particular examples:

First example: arrows in the plane

The first definition of a vector space is a fixed plane with arrows beginning at a specified point. This is the explanation of forces or speeds in physics. Given the two flips, v and w, there is one diagonal arrow in the parallelogram of the two arrows, which also begins at the origin. In the case of two arrows on the same side, the sum of this new flat is the arrow with a length that corresponds to the total or difference in length, depending on whether or not the arrows have the same direction. This flat is the total of both arrows and the length of the arrow. [1] Another operation that can be performed with arrows is the scaling process. Given any positive number as the arrow that is dilated or shrunk in the same direction as v, is called the multiplication of v by a, because of its length. It's referred to as av. Av is defined, if an is negative, as the flew pointing the other way.

The following shows a few examples: if a = 2, the resulting vector aw has the same direction as w, but is stretched to the double length of w (right image below). Equivalently, 2w is the sum w + w. Moreover, (−1)v = −v has the opposite direction and the same length as v (blue vector pointing down in the right image).



Second example: ordered pairs of numbers

The pairs of real numbers x and y are a second key example of a vector space. (The sequence is important for the components x and y, so that such a pair has been often referred to as the ordered pair.) The sum of two such pairs and multiplication of a pair with a number is defined as follows:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and

$$a(x, y) = (ax, ay).$$

A vector space over a field F is a set V together with two operations that satisfy the eight axioms listed below. In the following, V × V denotes the Cartesian product of V with itself, and → denotes a mapping from one set to another.

- ► The first operation, called vector addition or simply addition + : V × V → V, takes any two vectors v and w and assigns to them a third vector which is commonly written as v + w, and called the sum of these two vectors. (The resultant vector is also an element of the set V.)

- ► The second operation, called scalar multiplication · : F × V → V, takes any scalar a and any vector v and gives another vector av. (Similarly, the vector AV is a V set unit. It should not be confused with the scalar product, also known as the internal product or point product, which has an additional structure in some unique vector spaces but does not exist in all of them. Scalar multiplication is a scalar multiplication of a vector; the other means two vectors which generate a scalar.)

Elements of *V* are commonly called vectors. Elements of *F* are commonly called scalars. Common symbols for denoting vector spaces include *U*, *V* and *W*.[1]

The field is the field of actual numbers in the two examples above, and the set of vectors is composed of planar arrows with fixed starting point and pairs of real numbers.

The set *V* and the add-in / multiplication operations must comply with several criteria called axioms to be considered a vector space[2]. These are shown in the following table, with u, v and w indicating arbitrary *V* vectors and b and a denoting scalar of *F*. Indeed, the result of addition of two ordered pairs (as in the second example above) does not depend on the order of the summands:

$$(x_{\mathbf{v}}, y_{\mathbf{v}}) + (x_{\mathbf{w}}, y_{\mathbf{w}}) = (x_{\mathbf{w}}, y_{\mathbf{w}}) + (x_{\mathbf{v}}, y_{\mathbf{v}}).$$

In the geometric example, v + w = w + v as the vector’s space total parallelogram is not in the order of the vectores. v + w = w + v. In both cases, all other axioms can be tested similarly. Thus, the description combines these two and many more examples in one notion of vector space by disregarding the unique existence of the particular form of vector space.

Subtraction of two vectors and division by a (non-zero) scalar can be defined as

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$
$$\frac{\mathbf{v}}{a} = \frac{1}{a}\mathbf{v}$$

The vector space is called a true vector space if the scalar F is the real *R* number. The vector space is called complex vector space if the scalar field is the complex *C* number. The two cases in the engineering industry are most common. The overall vector space concept requires the scalar elements to be part of a fixed *F* field. The word F-vector space or vector space over F is then defined. A field is basically a set of numbers with operations to add, deduct, multiply and divide. Rational numbers, for instance, form a field.

There is no concept of proximity, angles or lengths, as opposed to the vector intuition in the plane and higher dimensional situations. Special forms of vector spaces are implemented for such matters; see § Vector spaces for more information below with additional structures.

Vector spaces

A vector space is a non-empty set V, whose objects are referred to as vectors, fitted with two operations called addition and scalar multiplication: there are uniform vectors u + v and cu in V of each of the two vectors u, v, and a scalar c, to satisfy the following properties.

1. $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$,
2. $(\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w} = \boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w})$,
3. There is a vector $\mathbf{0}$, called the **zero vector**, such that $\boldsymbol{u} + \mathbf{0} = \boldsymbol{u}$,
4. For any vector $\boldsymbol{u}$ there is a vector $-\boldsymbol{u}$ such that $\boldsymbol{u} + (-\boldsymbol{u}) = \mathbf{0}$;
5. $c(\boldsymbol{u} + \boldsymbol{v}) = c\boldsymbol{u} + c\boldsymbol{v}$,
6. $(c + d)\boldsymbol{u} = c\boldsymbol{u} + d\boldsymbol{u}$,
7. $c(d\boldsymbol{u}) = (cd)\boldsymbol{u}$,
8. $1\boldsymbol{u} = \boldsymbol{u}$.

By definition of vector space it is easy to see that for any vector u and scalar c,

$$0u = 0,\ c0 = 0,\ -u = (-1)u.$$

For instance,

$$\begin{aligned} 0\boldsymbol{u} &\overset{(3)}{=} 0\boldsymbol{u} + \mathbf{0} \overset{(4)}{=} 0\boldsymbol{u} + (0\boldsymbol{u} + (-0\boldsymbol{u})) \overset{(2)}{=} (0\boldsymbol{u} + 0\boldsymbol{u}) + (-0\boldsymbol{u}) \\ &\overset{(6)}{=} (0+0)\boldsymbol{u} + (-0\boldsymbol{u}) = 0\boldsymbol{u} + (-0\boldsymbol{u}) \overset{(4)}{=} \mathbf{0}; \\ c\mathbf{0} &= c(0\boldsymbol{u}) \overset{(7)}{=} (c0)\boldsymbol{u} = 0\boldsymbol{u} = \mathbf{0}; \\ -\boldsymbol{u} &= -\boldsymbol{u} + \mathbf{0} = -\boldsymbol{u} + (1-1)\boldsymbol{u} = -\boldsymbol{u} + \boldsymbol{u} + (-1)\boldsymbol{u} = \mathbf{0} + (-1)\boldsymbol{u} = (-1)\boldsymbol{u}. \end{aligned}$$

Example 1.

(a) The Euclidean space R n is a vector space under the ordinary addition and scalar multiplication.

(b) The set Pn of all polynomials of degree less than or equal to n is a vector space under the ordinary addition and scalar multiplication of polynomials.

(c) The set M(m, n) of all m × n matrices is a vector space under the ordinary addition and scalar multiplication of matrices.

(d) The set C[a, b] of all continuous functions on the closed interval [a, b] is a vector space under the ordinary addition and scalar multiplication of functions.

Example 2.

(a) For a vector space V, the set {0} of the zero vector and the whole space V are subspaces of V; they are called the trivial subspaces of V.

(b) For an m × n matrix A, the set of solutions of the linear system Ax = 0 is a subspace of R^n. However, if b 6= 0, the set of solutions of the system Ax = b is not a subspace of R^n.

(c) For any vectors v_1, v_2, . . . , vk in R^n, the span Span {v_1, v_2, . . . , vk} is a subspace of R n.

(d) For any linear transformation T : $R^n \rightarrow R^m$, the image

$$T(\mathbb{R}^n) = \{T(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n\}$$

of T is a subspace of R^m, and the inverse image

$$T^{-1}(\mathbf{0}) = \{\boldsymbol{x} \in \mathbb{R}^n : T(\boldsymbol{x}) = \mathbf{0}\}$$

is a subspace of R^n.

Alternative formulations and elementary consequences

Scalar multiplications are vector additions and scalar multiplication, which satisfy the closure properties: u + v and av are in V for all an in F, and u, v in V.

In the abstract algebra language, the first four axioms are analogous to the fact that the vector set should be an abelian group. The other axioms include a structure of this category with an F-module. In other words, the ring homomorphism f from field F into the vector group endomorphism ring. The scalar multiplication av, then (f(a))(v) is defined[6].

The vector space axioms have some direct implications. Some come from the basic group principle, applied to the vector group additive. For example, the 0 vector of 0 is unique and the reverse additive v of any vector v is unique. Additional properties follow those for scalar multiplication, for example av is equal to 0 if and only if the distributive law is equal to 0 or v.

Complex numbers and other field extensions

The set of complex numbers C, that is, numbers that can be written in the form x + iy for real numbers x and y where i is the imaginary unit, form a vector space over the reals with the usual addition and multiplication: $(x + iy) + (a + ib) = (x + a) + i(y + b)$ and $c \cdot (x + iy) = (c \cdot x) + i(c \cdot y)$ for real numbers x, y, a, b and c. The different axioms of a vector space are based on the assumption that the same laws apply to complex arithmetic numbers.

Indeed, the definition of a complex number (i.e., isomorphically) is basically the same for the vector space of the ordained actual number pairs described previously: when we consider the complex number x + iy to be the ordered pair (x , y) in the complex plane, we can see that there is a direct connexion between the additions and the scale-producing rules of the above case.

Field extensions typically provide another class of vector space examples, in algebra and algebraic numeric theory in particular; field F which contains a field E is an E-vector space, due to the multiplication and the addition of F. Field extensions are given. For example, the complex numbers are a vector space over R, and the field extension $\mathbb{Q}(i\sqrt{5})$ is a vector space over Q.

Function spaces

Functions from any fixed set Ω to a field F also form vector spaces, by performing addition and scalar multiplication point wise. That is, the sum of two functions f and g is the function $(f + g)$ given by

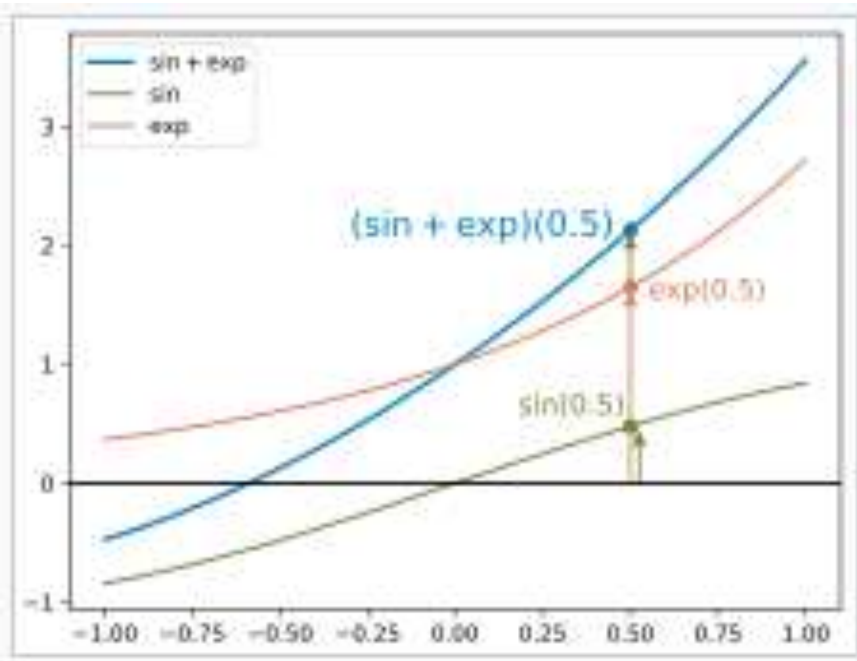
$$(f+g)(w) = f(w) + g(w),$$

and similarly for multiplication. In certain geometric situations such functional spaces occur when a ren is the actual line, an interval, or other sub-sets of R. Various principles in topology and analysis, such as continuity, integrability or differentiability, are well preserved as regards linearity: the sums and scalar multiples of functions with this sort of property still have the property [17]. The methods of functional analysis, see below, are discussed in more detail. Algebraic constraints also yield vector spaces: the vector space F[x] is given by polynomial functions:

$$f(x) = r_0 + r_1 x + \ldots + r_{n-1} x^{n-1} + r_n x^n,$$

where the coefficients

$$r_0, \ldots, r_n \text{ are in } F.$$



Addition of functions: The sum of the sine and the exponential function

Linear equations

Main articles: Linear equation, linear differential equation, and Systems of linear equations

Systems of homogeneous linear equations are closely tied to vector spaces. For example, the solutions of

$$a + 3b + \quad c = 0$$
$$4a + 2b + 2c = 0$$

are given by triples with arbitrary a, b = a/2, and c = −5a/2. The vector space is generated by the sums of these triples and scalar multiples; they are both solutions, and yet satisfy the same proportion of the three variables. Matrices can be used for condensing several linear equations into a single vector equation as above, namely

$$A\mathbf{x} = \mathbf{0},$$

Where

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 2 & 2 \end{bmatrix}$$

x is a vectour (a, b , c), Ax denotes the sum of the matrix and 0 = (0, 0) is the zero vector. Ax is the matrix of the equations. The solutions of uniform linear differential equations, in a similar way, form vector spaces. For example,

$$f''(x) + 2f'(x) + f(x) = 0$$

yields

$$f(x) = a\,e^{-x} + bx\,e^{-x},$$

where a and b are arbitrary constants, and ex is the natural exponential function.

Linear maps and matrices

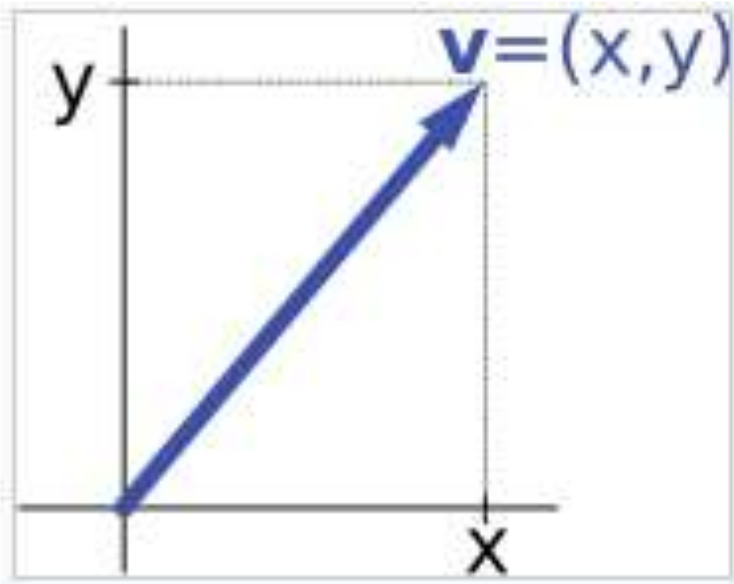
In linear map or linear transformation can express the relationship of two vector spaces. They represent the space structure vector, that is to say, maintain amounts and scalar multiplication:

$$f(\mathbf{v} + \mathbf{w}) = f(\mathbf{v}) + f(\mathbf{w}) \text{ and } f(a \cdot \mathbf{v}) = a \cdot f(\mathbf{v})$$

for all v and *w* in *V*, all a in F.

An isomorphism is a linear map f: V → W such that there exists an inverse map g: W → V, which is a map such that the two possible compositions f ∘ g: W → W and g ∘ f: V → V are identity maps. Equivalently, f is both one-to-one (injective) and onto (surjective). If an isomorphism occurs between V and W, both spaces would be isomorphic and are basically identical as vector spaces since all identities containing V are transported via V to similar spaces in W and vice versa via g.

For instance, the input "plane arrows" and "ordered numbers pairs" are isomorphic: the x- and y-component of the arrow can be taken into consideration by the planned arrow v which starts at the beginning of the (fixed-) co-ordinate system in an ordered pair. In comparison, if a pair (x , y) is seen, the arrown is x to the right and y up (down, where y is negative) to x to the left. Linear maps V → W between two vector spaces form a vector space HomF(V, W), also denoted L(V, W). The space of linear maps from V to F is called the dual vector space, denoted V∗.Via the injective natural map V → V∗∗, any vector space can be embedded into its bidual; the map is an isomorphism if and only if the space is finite-dimensional.



Describing an arrow vector v by its coordinates x and y yields an isomorphism of vector spaces

Once a basis of V is chosen, linear maps f : V → W are completely determined by specifying the images of the basis vectors, because any element of V is expressed uniquely as a linear combination of them. If dim V = dim W, a 1-to-1 correspondence between fixed bases of V and W gives rise to a linear map that maps any basis element of V to the corresponding basis element of W. It is an isomorphism, by its very definition. Therefore, two vector spaces are isomorphic if their dimensions agree and vice versa. Another way to express this is that any vector space is completely classified (up to isomorphism) by its dimension, a single number. In particular, any n-dimensional F-vector space V is isomorphic to Fn. There is, however, no "canonical" or preferred isomorphism; actually an isomorphism φ : Fn → V is equivalent to the choice of a basis of V, by mapping the standard basis of Fn to V, via φ.

CONCLUSION

Vector spaces stem from affine geometry, via the introduction of coordinates in the plane or three-dimensional space. By defining solutions to an equation of two variables with points on a plane curve, mathematicians René Descartes and Pierre de Fermat created analytical geometry. Vectors were checked when Argand and Hamilton introduced complex numbers and the latter started quaternions. These are R2 and R4 elements; they are handled using linear combinations and specified linear equation systems. There are definitions of linear autonomy and dimension as well as scalar products. The 1844 work of Grassmann actually goes beyond the vectors, because his careful reproduction often leads him to what today are referred to as algebras. The first to give a modern description of vector spaces and linear maps was the Italian mathematician Peano. The construction of functional spaces by Henri Lebesgue is a significant advancement of vector spaces. Algebra and the new area of functional analysis began interacting at the time , especially with key concepts such as P-integrable spaces and Hilbert spaces. The first experiments on infinite dimensional vector spaces were also carried out at this time.

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