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A Research on the Theory of Integration on Locally Compact Spaces: A Case Study of Generalized Riemann Integral

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Abstract – We extend the fundamental results on the hypothesis of the generalized Kiemann integral to the setting of limited or locally limited measures on locally compact second countable Hausdorff spaces. The correspondence between Borel gauges on X and persevering valuations on the upper space UX offers rise to a topological embeddings between the space of locally limited measures and locally limited reliable valuations, both contributed with the Scott topology. We assemble an approximating chain of basic valuations on the upper space of a locally compact space, whose least upper bound is the given locally limited measure. The generalized Kiemann integral is portrayed for limited capacities with respect to both limited and locally limited measures.

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INTRODUCTION

A central relationship between space hypothesis and some essential pieces of science has been set up, giving rising, explicitly, to a novel computational approach to manage measure hypothesis and integration. An area theoretic structure for measure and integration has been given, by showing that any limited Borel measure on a compact measurement space A can be gotten as the least upper bound of basic valuations (measures) on the upper space IX. the course of action of non-void compact subsets of An orchestrated by switch consolidation.

Basic valuations approximating the given measure expect the activity of portions in Riemann integration and are used to improve approximations of the integral of a limited genuine esteemed limit on a compact measurement space. As such, as opposed to approximating the limit with basic capacities as is done in Lebesgue hypothesis, the measure is approximated with basic measures. This idea prompts another thought of integration, called generalized Riemann integration. R-integration for short, equivalent in soul anyway more expansive than Riemann integration.

All the principal eventual outcomes of the hypothesis of Riemann integration can be connected with this setting. For example that a limited genuine esteemed limit on a compact measurement space is R-Integrable with respect to a limited Borel measure if and just if its game plan of discontinuities has measure zero and that if the limit is R-integrable, by then it is moreover Lebesgue integrable and the two integrals orchestrate.

The area theoretic speculation of tin* Riemann integral works all around for integration of capacities with respect to Borel gauges on Polish spaces (topologically complete recognizable metrizable spaces) which consolidate locally compact second countable spaces. Here, one moreover deals with the area of the limit rather than its range. Cabin now one goes past the idea of packages and uses limited covers by open subsets to offer approximations to the measure. These approximations give generalized upper and lower wholes with which we describe the integral.

THE INTEGRAL

Give T a chance to be a locally compact isolated space. (Review "isolated" is proportional to "HausdorfL") An isolated space is locally compact if each point has a compact neighborhood. Let C(T) be the arrangement of all nonstop, complex valued functions f characterized on T. Review that the help of $f \in C(T)$ is the closure of the set $\{x : f(x) \neq 0\}$ *. We define $C_0(T)$ to be the subspace of C(T) comprising of functions of compact help. On $C_0(T)$ we can characterize the standard $\|f\| = \sup\{|f(x)| : x \in T\}$. We present further the spaces at least $C_0^r(T), C_0^+(T)$ quickly C_0^r, C_0^+:

(1) $C_0^r(T)$ is all real valued functions in $C_0(T)$.

(2) $C_0^+(T)$ is all nonnegative functions in $C_0(T)$.

DEFINITION 1. *An integral I is a linear functional on* $C_0(T)$ *such that* $I(f) \geq 0$ *whenever* $f \in C_0^+(T)$.

It follows that *1(f)* is real if $f \in C_0^r$ (because *f* can be written as $f = f^+ - f^-$ where $f^\pm \in C_0^+$). Also if $f \geq g$ then $I(f) \geq I(g)$ for any $f, g \in C^r$.

Example: The Riemann integral defined on $\mathbb{R}$ is an example of an integral because every function $f \in C_0(\mathbb{R})$ is Riemann integrable and of course a nonnegative function has a nonnegative integral.

Lemma 1. $|I(f)| \leq I(|f|)$, *for all* $f \in C_0$.

PROOF. As a complex number $I(f) = re^{i\theta}$ where $r = |I(f)|$. If we expand $e^{-i\theta} f = u + iv$ where $u, v \in C_0^r$ then we have

$$\begin{aligned} r = e^{-i\theta} I(f) = I(e^{-i\theta} f) &= I(u + iv) \\ &= I(u) + iI(v) \\ &= I(u) \leq I(|u|) \leq I(|u + iv|) = I(|f|) \end{aligned}$$

which is precisely $|I(f)| \leq I(|f|)$.

Our objective is to extend the integral to a wider class of functions. The expansion methodology underneath, whenever applied to the Riemann integral, would prompt the Lebesgue integral.

LEMMA 2. In the event that Q is a compact, subset of T then there is a constant $C_Q > 0$ so that for every one of the $f \in C_0(Q)$

$$|I(f)| \leq C_Q \|f\|$$

PROOF. Urysohn's Lemma applies (Theorem ??) and enables us to infer that there exists a capacity $g \in C_0^+(T)$ so that $g(x) = 1$ if $x \in Q$. (It was indicated that a compact Hausdorff space is typical in the Example earlier Theorem ?? with the goal that Urysohn's Lemma applies in a compact neighborhood of Q.)

Therefore for all $f \in C_0(Q)$

$$I(f) = I(gf) \leq I(\|f\| g) = \|f\| I(g) \equiv C_Q \|f\|$$

Where $C_Q = I(g)$. □

PROPOSITION 1. *Suppose that* $X \subseteq C_0^+(T)$ *is directed downwards (that is for every* $f, g \in X$ *there is* $h \in X$ *so that* $h \leq f$ *and* $h \leq g$*). Suppose further that, for each* $t \in T$

$$\inf_{f \in X} f(t) = 0$$

Then, for all $\epsilon > 0$, *there is* $f_\epsilon \in X$ *so that* $\|f_\epsilon\| < \epsilon$ *and therefore*

$$\inf_{f \in X} I(f) = 0$$

PROOF. For $\epsilon > 0$, and $f \in X$ define $A_\epsilon(f) = \{t : f(t) \geq \epsilon\}$. If $f \geq g$ then $A_\epsilon(f) \supseteq A_\epsilon(g)$. Because inf $_{f \in X} f(t) = 0 \; \cap_{f \in X} A_\epsilon(f) = \emptyset$.

For any $f \in X$, $A_\epsilon(f)$ is compact and so there must be finitely many *f*, $f_2, \ldots, f_n$ so that $\cap_{1 \leq k \leq n} A_\epsilon(f_k) = \emptyset$. Because *X* is directed downward, there is $f_\epsilon \in X$ so that $A_\epsilon(f_\epsilon) = \emptyset$ or in other words $|f_\epsilon(t)| < \epsilon$.

(Indeed given $f \in X$ we can choose $f_\epsilon \in X$ so that $f_\epsilon \leq f$ and so $\inf_{f \in X} I(f) = 0$, by the preceding Lemma.

INTEGRATION ON LOCALLY COMPACT SPACES GENERATED BY POSITIVE LINEAR FUNCTIONALS

A non-empty family *V* of sets of an abstract space *X* is called a *prering* if the following condition is satisfied: if $A, B \in V$ then $A \cap B \in V$ and there exists disjoint sets $C_1, \cdots, C_k \in V$ such that $A \backslash B = C_1 \cup \cdots \cup C_k$.

A function μ from a prering *V* into a Banach space *Z* is called a vector volume if it satisfies the following condition: for every countable family of disjoint sets $A_t \in V (t \in T)$ such that

(a) $$A = \bigcup_T A_t \in V$$

we have $\mu(A) = \sum_T \mu(A_t)$, where the last sum is convergent absolutely and the variation of the function μ, that is, the function

$$|\mu|(A) = sup\left\{\sum_T |\mu(A_t)|\right\}$$

is finite for every set $A \in V$, where the supremum is taken over all possible decompositions of the set *A* into the form (a). A volume is called positive if it takes on only non-negative values. If μ is a volume then its variation $|\mu|$ is a positive volume.

If v is a volume on a prering *V* of subsets of a space *X* then the triple (X, V, v) is called a volume space.

Let *R* be the space of reals and *Y, Z, W* be Banach spaces. Denote by *X* the space of all bilinear continuous operators u from the space *Y* x *Z* into the space *W*. Norms of elements in the spaces *Y, Y', Z, W, U* will be denoted by $|\ |$.

In the paper Bogdanowicz, W. M.: (2005) has been presented an approach to the theory of the space $L(v, Y)$ of Lebesgue-Bochner summable functions generated by a positive volume v. The construction was not based on measure or on measurable functions. It allowed us to prove the basic structure theorems of the space of summable functions and at the same time to develop the theory of an integral of the formwhere *u* denotes a bilinear continuous operator from $\int u(f, d\mu), U = L(Y, Z; W), f \in L(v, Y),$ and μ is a finitely additive function from the prering *V* to the Banach space *Z,* dominated by the volume cv for some constant $c,$ that is, such that the estimation holds $|\mu(A)| \leq cv(A)$ for all $A \in V.$

The construction of the theory of Lebesgue-Bochner measurable functions and the theory of measure corresponding to the approach of Bogdanowicz, W. M.: (2005) has been developed, This approach permitted us to simplify in Bogdanowicz, W. M.: (2005) the construction and the theory of integration on locally compact spaces.

In the paper Bogdanowicz, W. M.: (2005) has been presented an approach to the theory of integration generated by a positive linear functional defined on any linear lattice of real-valued functions. The approach was based only on the results of Bogdanowicz, W. M.: (2005),

Using the results of Bogdanowicz, W. M.: (2005) we shall show in Bogdanowicz, W. M.: (2005) of this paper how one may develop the theory of integration on locally compact spaces generated by positive linear functionals defined on the space C_0 of all continuous functions *f* with compact support from a locally compact space *X* into the space of reals *R.*

We will say that the set $A \subset X$ is bounded if its closure is compact. The family V_1 of all sets of the form $A = G_1 \backslash G_2,$ where G_i are open, bounded sets, forms a prering which will be called the *Bor el prering.*

The family *V* of all sets of the form $A = G_1 \backslash G_2,$ where G_i are open bounded F_σ sets, forms a prering which will be called the *Baire prering.*

The smallest sigma-ring containing the Baire prering or the Borel prering will be called respectively the *Baire or the Borel ring.* It is easy to see that the Borel ring is the smallest sigma- ring containing all bounded open sets, and the Baire ring is the smallest sigma-ring containing all open bounded F_σ sigma sets.

A real-valued function v on a family of sets *V* of a topological space *X* is called regular if the following conditions are satisfied $v(A) = sup\{v(E) : \bar{E} \subset A, E \in V\}$ and $v(A) = inf\{v(E) : A \subset int\, E, E \in V\}$ for all sets $A \in V.$

A positive volume or a positive measure defined on the Borel prering or the Borel ring, respectively, is called *Borel volume or Borel measure*, respectively, if it is regular.

A positive volume or a positive measure defined on the Baire prering or the Baire ring, respectively, is called the *Baire volume or the Baire measure.* It is easy to prove that every Baire volume and therefore every Baire measure is regular.

LOCALLY FINITE MEASURES ON A LOCALLY COMPACT SPACE

All through the paper, X will mean a second countable locally compact Hausdorff space. We will utilize the decay $X = \bigcup_{i \in \mathbb{N}} X_i,$ where $\langle X_i \rangle_{i \in \mathbb{N}}$ is an expanding arrangement of moderately compact open subsets of X to such an extent that $\overline{X_i} \subseteq X_{i+1}.$ We start with certain definitions:

Definition 1 A Borel measure μ on a locally compact Hausdorff space is locally finite if $\mu(C) < \infty$ for all compact $C \subseteq X.$ $M^\ell(X)$ will mean the arrangement of locally finite measures on X. The arrangement of measures nagged by one and the arrangement of standardized measures are meant individually by $M(X)$ and $M^1(X).$

We review from A. Edalat.(2005) that the upper space UX of a topological space is the arrangement of all non-void compact subsets of X . with the base of the upper topology given by the sets

$$\Box a = \{C \in UX \mid C \subseteq a\},$$

where $a \in \Omega(X).$ When X is a second countable locally compact Hausdorff space, at that point the upper space VX of X is a ω- nonstop depo and the Scott topology of $(UX, \supseteq)$ matches with the upper topology. The lub of a coordinated subset is the crossing point and $A \ll B$ iff B is contained the inside of X. The singleton map $s : X \to UX$ with $s(x) = \{x\}$ is a topological implanting onto the arrangement of maximal components of UX.

It was indicated that the guide

$$\begin{array}{ccc} M(X) & \to & P(UX) \\ \mu & \mapsto & \mu \circ s^{-1} \end{array}$$

is an infusion into the arrangement of maximal components of P(U A) and it was guessed that its picture is the arrangement of maximal components. This guess was later demonstrated by Lawson in a progressively broad setting. The consistent dcpo U A doesn't really have a base component. In this way, so as to consider standardized valuations, we will append a base component $\perp = X$ and indicate the dcpo with base subsequently acquired with $(UX)_\perp.$ Then the injective guide $\mu \mapsto \mu \circ s^{-1} : M^1(X) \to P^1(UX)_\perp$ is onto the set of maximal components of $P^1(UX)_\perp.$ Here, we will demonstrate a balanced correspondence between locally finite Borel measures on An and locally finite persistent valuations on the upper space bolstered in $s(X).$

Proposition 1 Let $s : X \to UX$ be Like singleton map. Then the guide

$$\begin{array}{rccc} e : & M^t(X) & \to & P^t(UX) \\ & \mu & \mapsto & \mu \circ s^{-1} \end{array}$$

is very much characterized.

Before demonstrating the above proposition we need the accompanying lemmas, interfacing the way-underneath connection on the upper space UX of X with the one on X

Lemma 1 Led $\{O_i : i \in I\}$ be a coordinated family in $\Omega(X)$. At that point

$$\square(\bigcup_{i \in I} O_i) = \bigcup_{i \in I} \square O_i.$$

Proof: The consideration from right to left inconsequentially holds. For the opposite, accept that. C is a non-empty compact subset of $\bigcup_{i \in I} O_i$. By compactness, C has a finite subcover, and therefore, since the family of opens is directed, there exists $i \in I$ such that C is a subset of O_i. □

Lemma 2 Let V be an open set in the Scoff topology of UX. Then $V \ll UX$ in $\Omega(UX)$ if and only if $s^{-1}(V) \ll X$ in $\Omega(X)$.

Proof:

⇒: Suppose $X \subseteq \bigcup_{i \in I} O_i$ where the right-hand side is a directed union and $O_i \in \Omega(X)$.

Then, by lemma 1. we have $UX = \square X \subseteq \bigcup_{i \in I} \square O_i$. Since $V \ll UX$ in $\Omega(UX)$, there exists $i \in I$ such that $V \subseteq \square O_i$, and therefore $s^{-1}(V) \subseteq O_i$, thus proving the claim.

⇐: Suppose $UX \subseteq \bigcup_{i \in I} O_i$, where the right-hand side is the union of a directed family of opens of the upper space. Since by hypothesis $s^{-1}(V) \ll X$ and X is a locally compact space, there exists a compact set $C \subseteq X$ such that $s^{-1}(V) \subseteq C \subseteq X$. Since $C \in UX$, there exists $i \in I$ such that $C \in O_i$ and therefore $\uparrow C \subseteq O_i$ as O_i is open in the Scott topology of *U*X and thus an upper set with respect to reverse inclusion. This implies $V \subseteq \uparrow C$. For. If $K \in V$ and $x \in K$, then $K \sqsubseteq \{x\}$ and. hence. $\{x\} \in V$ since *V* is upward closed. Thus $x \in s^{-1}(V) \subseteq C$, i.e. $K \subseteq C$. Therefore $V \subseteq \uparrow C \subseteq O_i$, i.e.. $V \ll UX$.

Proof of proposition 3.2: Let μ he a locally finite Borel measure on X. Then it is immediate to verify that $\mu \circ s^{-1}$ satisfies since μ is a measure and s^{-1} preserves (directed) unions and intersections. The continuous valuation $\mu \circ s^{-1}$ is locally finite since, if $V \ll UX$, then, by lemma 2, $s^{-1}(V) \ll X$. Since *X* is locally compact, there exists a compact subset *K* of *X* such that $s^{-1}(V) \subseteq K$. Therefore, by monotonicity of μ, $\mu \circ s^{-1}(V) \leq \mu(K)$, and by the assumption on local finiteness of μ the conclusion follows.

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