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Simultaneous Identification of Two Time Independent Coefficients in a Nonlinear Phase Field System

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Abstract – In the mathematical literature, the free boundary problems arising from phase transitions have been studied for over a century. Most of the work is concerned with the classical Stefan problem which incorporates the physics of latent heat and heat diffusion in a homogeneous medium. It should be recalled that the phase field models were first introduced and recently several authors redis-cussed the problem and also improved the results from the thermo dynamical point of view; see, for instance, the work for an exhaustive explanation of the underlying physics and in fact these works provide an extension of the enthalpy method for the Stefan problem with the advantage of making it possible to describe some rather fine physical phenomena which can take place during fusion-solidification processes. Moreover, in the recent years, the study of several variants of the model has been done and interesting results have been obtained in the directions of existence and regularity of solutions as well as of their dependence on the physical parameters and one can see and for the dynamical controllability of phase field models with one and two control forces respectively.

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In this work, we consider the phase field model describing the phase transitions between two states, for example, solid or liquid, in pure material

$$\left.\begin{aligned}
&u_t + \ell v_t - u_{xx} + a(x)u = 0,\ (x,t) \in \Omega_T = I \times (0,T],\\
&v_t - v_{xx} + b(x)f(v) + c(x)u = 0,\ (x,t) \in \Omega_T,\\
&u(x,0) = \phi(x),\ v(x,0) = \varphi(x),\ x \in I,\\
&u(0,t) = u(1,t) = v(0,t) = v(1,t) = 0,\ t \in (0,T],
\end{aligned}\right\}$$

Where I = (0, 1). The solution u denotes the temperature distribution of a material which occupies the region ΩT and can be in either of the two phases, solid or liquid (if the melting temperature is taken to be zero) and the smooth function v is called the phase field function. The interface is defined implicitly as the set of points for which v vanishes (see [16, 39]), where ` is the latent heat and the initial data φ(x) and ϕ(x), depending only on spatial variable x, are sufficiently regular. The coefficients a(x), b(x), c(x) are assumed to be sufficiently smooth and shall be kept independent of time t and the nonlinear function f is a polynomial or rational function of v which is sufficiently smooth. The problem of solving these equations on some space time domain provided that the coefficients, boundary conditions and initial condition is called the direct problem. These types of models have been analyzed in depth mathematically by several authors. The main objective of this chapter is to obtain the stability estimate for an inverse problem of determining two spatial dependent coefficients a(x) and b(x) in the nonlinear phase field system from the following final time over determination data

$$u(x,T) = m(x), \quad v(x,T) = n(x), \quad x \in I, \tag{5.1}$$

where the functions m(x) and n(x) are given and satisfy the homogeneous Dirichlet boundary conditions. However, over the past several decades, various methods have been employed to solve the inverse problems for phase field system. For example, established the stability estimate for the identification of two time independent coefficients in the linear phase field model from the arbitrary sub domain observation using Carleman estimate and discussed the inverse problem of retrieving spatially varying diffusive coefficient in the linear phase field system from single measurement data at arbitrary sub domain in analyzed the phase field model for solidification with constant coefficient for the stability of construction of the source term using optimal control framework by considering the weighted quadratic cost function. Different from the above mentioned work, we establish the stability estimate for the simultaneous reconstruction of two spatially varying parameters in the nonlinear phase field system. The optimal control technique is the key step in establishing the stability estimate.

OPTIMAL CONTROL

Let us first convert the identification problem into an optimal control problem based on the final time over specified output data and then prove the existence of the minimize of the cost functional based on the assumption of the admissible parameter. The following assumption on the parameters and the nonlinear functional are essential to prove the stability result

Assumption 5.2.1 For α > 0, we assume that the coefficients $a(x), b(x), c(x) \in$

$C^{\alpha}(\bar{I})$, *the initial data* $\phi(x), \varphi(x) \geq 0,\ \phi(x), \varphi(x) \in C^{2,\alpha}(\bar{I})$ and the final time observation $m(x), n(x) \in L^2(I)$.

$$\text{Let } \Psi = \max_{x \in I} |\varphi(x)|.$$

We suppose that the function $f \in C^2[0, \Psi]$ satisfies

$$f(0) = 0, \quad f'(\cdot) > 0, \quad f''(\cdot) \leq 0 \quad \text{in } [0, \Psi]. \tag{5.2}$$

Physically it is very reasonable to search for the parameters a(x) and b(x) among all positive functions which are bounded below and bounded above by two roughly predicted fixed positive constants. Hence we define the admissible set M by

$$\mathcal{M} = \{a(x), b(x) | 0 < a_0 \leq a \leq a_1,\ 0 < b_0 \leq b \leq b_1,\ \nabla a, \nabla b \in L^2(I)\} \tag{5.3}$$

and the optimal control problem can be stated as follows: Find (¯a(x), ¯b(x)) ∈ M × M satisfying

$$\mathcal{J}(\bar{a}, \bar{b}) = \min_{a,b \in \mathcal{M}} \mathcal{J}(a, b), \tag{5.4}$$

$$\begin{aligned} \mathcal{J}(a, b) =& \frac{1}{2} \int_I \left(|u(x, T; a) - m(x)|^2 + |v(x, T; b) - n(x)|^2\right) \mathrm{d}x \\ &+ \frac{\wp}{2} \int_I \left(|\nabla a|^2 + |\nabla b|^2\right) \mathrm{d}x, \end{aligned} \tag{5.5}$$

and (u, v) is the solution of the system (5.1.1) for the coefficients a(x), b(x) ∈ M, a0, a1, b0 and b1 are given positive constants and $\wp$ is the regularization parameter. By the well-known Schauder's theory for parabolic equations, we can prove the following existence result (see [45, 47, 77, 90]).

Theorem 5.2.1. Let 0 < α < 1 and the coefficients a(x), b(x), c(x) ∈ C α (¯I). Then the system (5.1.1) has a unique solution

$$u(x, t), v(x, t) \in C^{2+\alpha, 1+\frac{\alpha}{2}}(\bar{\Omega}_T).$$

The following theorem proves the existence of the optimal control ¯a(x), ¯b(x) ∈ M minimizing the cost functional J (a, b).

Theorem 5.2.2. Suppose (u, v) is the solution of the system (5.1.1). Then there exists a minimizer a, ¯ ¯b ∈ M of J (a, b) such that

$$\mathcal{J}(\bar{a}, \bar{b}) = \min_{a,b \in \mathcal{M}} \mathcal{J}(a, b),$$

where the admissible set M and J (a, b) are the same as defined in (5.2.2) and (5.2.4).

Proof. We use the generalized in order to prove the existence of the solution of the optimal control problem. According to generalized Weierstrass theorem, we must show that the set M is closed, bounded and convex and the functional J is lower semi continuous. The bounded-ness and convex city of the admissible set M is obvious from the definition. Now we have to show that the closed-ness of the admissible set M. From the structure of the construction of the cost functional J (a, b), note that the function J (a, b) is nonnegative and thus it has the greatest lower bound. From the definition of the greatest lower bound, it implies that there exists a sequence (un, vn, an, bn) such that

$$\inf_{a,b \in \mathcal{M}} \mathcal{J}(a, b) \leq \mathcal{J}(a_n, b_n) \leq \min_{a,b \in \mathcal{M}} \mathcal{J}(a, b) + \frac{1}{n}, \quad n = 1, 2 \cdots$$

where {an, bn} are minimizing sequences of J(a, b) in M. From the cost functional, we observe that J(an, bn) ≤ C, from which we easily deduce that

$$\|\nabla a_n\|_{L^2(I)} + \|\nabla b_n\|_{L^2(I)} \leq C,$$

Thus, by the existence of classical solutions of parabolic equations, we have

$$\|u_n\|_{C^{\frac{1}{2}, \frac{1}{4}}(\bar{\Omega}_T)} + \|v_n\|_{C^{\frac{1}{2}, \frac{1}{4}}(\bar{\Omega}_T)} \leq C,$$

And, for any ωT b ΩT , we also get

$$\|u_n\|_{C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\omega_T)} + \|v_n\|_{C^{2+\frac{1}{2}, 1+\frac{1}{4}}(\omega_T)} \leq C. \tag{5.6}$$

The bounded-ness results (5.2.5) and (5.2.6) guarantee that there exists a subsequence $(u_{n_k}, v_{n_k}, a_{n_k}, b_{n_k})$ such that

$$\begin{aligned} (a_{n_k}, b_{n_k}) &\to (\bar{a}, \bar{b}) \in \left(C^{\frac{1}{2}}(I)\right)^2 \text{ uniformly on } \left(C^{\alpha}(\bar{I})\right)^2, \\ (u_{n_k}, v_{n_k}) &\to (\phi, \varphi) \text{ uniformly on } \left(C^{\alpha, \frac{\alpha}{2}}(\bar{\Omega}_T) \cap C^{2+\alpha,\ 1+\frac{\alpha}{2}}_{\text{loc}}(\Omega_T)\right)^2. \end{aligned}$$

Hence, replacing (u, v, a, b) in (5.1.1) by (*unk* , *vnk* , *ank* , *bnk*) and passing to the limit, we can see that (φ, ϕ, a, ¯ ¯b) satisfy the system (5.1.1). The closed-ness is proved by showing that the limit of every convergent sequence is an element of M. In order to show that the cost functional is lower semi continuous, consider

$$\int_I \left(|\nabla(a_{n_k} - \bar{a})|^2 + |\nabla(b_{n_k} - \bar{b})|^2\right) \mathrm{d}x \geq 0,$$

That is

$$\int_I \left(|\nabla a_{n_k}|^2 + |\nabla b_{n_k}|^2\right) \mathrm{d}x \geq 2\int_I \left(\nabla a_{n_k} \nabla \bar{a} + \nabla b_{n_k}\ \nabla \bar{b}\right) \mathrm{d}x - \int_I \left(|\nabla \bar{a}|^2 + |\nabla \bar{b}|^2\right) \mathrm{d}x.$$

It follows

$$\begin{aligned} \lim_{k\to\infty} \int_I \left(|\nabla a_{n_k}|^2 + |\nabla b_{n_k}|^2\right) \mathrm{d}x \\ &\geq 2 \lim_{k\to\infty} \int_I \left(\nabla a_{n_k} \nabla \bar{a} + \nabla b_{n_k}\ \nabla \bar{b}\right) \mathrm{d}x - \int_I \left(|\nabla \bar{a}|^2 + |\nabla \bar{b}|^2\right) \mathrm{d}x \\ &= \int_I \left(|\nabla \bar{a}|^2 + |\nabla \bar{b}|^2\right) \mathrm{d}x. \end{aligned}$$

dx. And so

From the above observations, we can easily deduce that the cost functional is lower semi continuous, that is

$$\min_{a,b\in\mathcal{M}} \mathcal{J}(a,b) \leq \mathcal{J}(\bar{a},\bar{b}) \leq \liminf_{n\to\infty} \mathcal{J}(a_n,b_n) = \min_{a,b\in\mathcal{M}} \mathcal{J}(a,b),$$

$$\mathcal{J}(\bar{a},\bar{b}) = \min_{a,b\in\mathcal{M}} \mathcal{J}(a,b);$$

Thus (¯a, ¯b) := (a, b) is an optimal solution of the optimal control problem (5.2.2)- (5.2.4). Hence the proof is complete.

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