Journal of Advances and Scholarly Researches in Allied Education
Vol. XV, Issue No. 1, April-2018, ISSN 2230-7540

Existence of Unique Common Fixed Point for Pairs of Mappings in Complete Metric Spaces

Anil Saini*

Assistant Professor of Mathematics, Pt. C. L. S. Govt. College, Sector – 14, Karnal, Haryana

Abstract – Existence and uniqueness of a fixed point is proved by Banach(1922), which is known as Banach contraction principle [1]. It is a very useful, simple, and classical tool in fixed point theory. Many authors have studied and extended this theorem in different ways. In this paper we also prove a new result concerning fixed point on two complete metric spaces. In this paper, we prove a common fixed point theorems for two pairs of maps in two complete metric spaces. These theorems are versions of many known results in metric spaces.

Key Words and Phrases – Complete Metric Spaces, Common Fixed Points, Mapping.

- X -

1. INTRODUCTION AND PRELIMINARIES

Banach (1922) proved a theorem which ensures the existence and uniqueness of a fixed point under appropriate conditions. His result is called Banach's fixed point theorem or the Banach contraction principle. This theorem is also applied to show the existence and uniqueness of the solutions of differential and integral equations and many other applied mathematics. Many authors have extended, generalized and improved Banach's fixed point theorem in different ways. Some fixed point theorems for two metric spaces have been proved by Brain Fisher [1], V. Popa [3], P.P. Murthy *et al* [4], R.K. Namdeo [5] and Luljeta Kikina *et al* [7]. Now our aim is to generalize and extend result of [3].

Definition 1.1 *Let (X, d) be metric space. A sequence* $\{x_n\} \in X$ *is said to be convergent to a point* $P \in X \Leftrightarrow \forall\ \varepsilon > 0\ \exists$ *a positive integer* $n_0(\varepsilon)$ *such that*

$$d(x_n, P) < \varepsilon \quad \forall\ n \geq n_0$$

Definition 1.2 *Let (X, d) be a metric space. A sequence* $\{x_n\} \subset X$ *is said to be Cauchy sequence if* $d(x_m, x_n) \to 0\ as\ m, n \to \infty$

Definition 1.3 *A metric space (X, d) is said to be complete if and only if Cauchy sequence in X converge to a point of X.*

The following theorem was proved by V. Popa [3].

Theorem 1.1 *Let (X, d) and (y, p) be complete metric spaces. If T: X → Y and S:Y→X satisfying the inequalities.*

$$\begin{aligned}\rho^2(T_x, TS_y) &\leq c_1 \max\{d(x,Sy)\rho(y,Tx),\\ &\quad d(x,Sy)\rho(y,TSy), \rho(y,Tx)\rho(y,TSy)\}\end{aligned} \tag{1.1}$$

$$\begin{aligned}d^2(Sy, STx) &\leq c_2 \max\{\rho(y,Tx)d(x,Sy),\\ &\quad \rho(y,Tx)d(x,STx), d(x,Sy)d(x,STx)\}\end{aligned} \tag{1.2}$$

$\forall\ x \in X$ and $y \in Y$, where $0 c_1, c_2 < 1$, then ST has a unique fixed point point $z \in X$ and TS has a unique fixed point $w \in Y$. Further, $Tz = w$ and $Sw = z$.

2. MAIN RESULT

Theorem 2.1 *Let (X, d_1) and (Y, d_2) be complete metric spaces. Let A, B: X → Y and C, D:Y→X satisfying the inequalities.*

$$\begin{aligned}d_1^2(Cy, DBx) &\leq c_1 \max\{d_2(y,Bx)d_1(x,Cy),\\ &\quad d_2(y,Bx)d_1(x,DBx),\\ &\quad d_1(x,Cy)d_1(x,DBx)\}\end{aligned} \tag{2.1}$$

$$\begin{aligned}d_2^2(Bx, ADy) &\leq c_2 \max\{d_1(x,Dy)d_2(y,Bx),\\ &\quad d_1(x,Dy)d_2(y,ADy),\\ &\quad d_2(y,Bx)d_2(y,ADy)\}\end{aligned} \tag{2.2}$$

$\forall\ x \in X$ and $y \in Y$ where $0 \leq c_1, c_2 < 1$. If one of mappings *A, B, C,* and *D* is continuous then *CA* and *DB* have a unique common fixed point $z \in X$ and *BC* and *AD* have a unique common fixed point $w \in Y$. Further $Az = Bz = w$ and $Cw = Dw = z$

Proof. Let *x* be an arbirary point in *X*. Let

$$Ax = y_1, Cy_1 = x_1, Bx_1 = y_2,$$

$$Dy_2 = x_2, Ax_2 = y_3$$

and in general let,

$$cy_{n-1} = x_{n-1}, Bx_{n-1} = y_n,$$

$$Dy_n = x_n, Ax_n = y_{n+1} \text{for} n = 1,2, \ldots$$

Using inequality (2.1), we get,

$$d_1^2(x_n, x_{n+1}) = d_1^2(Cy_n, DBx_n)$$
$$\leq c_1 \max\{d_2(y_n, Bx_n)d_1(x_n, Cy_n),$$
$$d_2(y_n, Bx_n)d_1(x_n, DBx_n),$$
$$d_1(x_n, Cy_n)d_1(x_n, DBx_n)\}$$

$$d_1^2(x_n, x_{n+1}) \leq c_1 \max\{d_2(y_n, y_{n+1})d_1(x_n, x_n),$$
$$d_2(y_n, y_{n+1})d_1(x_n, x_{n+1}),$$
$$d_1(x_n, x_n)d_1(x_n, x_{n+1})\}$$

$$d_1^2(x_n, x_{n+1}) \leq c_1 \max\{0, d_2(y_n, y_{n+1})d_1(x_n, x_{n+1}), 0\}$$
$$d_1^2(x_n, x_{n+1}) \leq c_1 d_2(y_n, y_{n+1})d_1(x_n, x_{n+1})$$
$$\Rightarrow d_1(x_n, x_{n+1}) \leq c_1 d_2(y_n, y_{n+1})$$

Now

$$d_2^2(y_n, y_{n+1}) = d_2^2(Bx_{n-1}, ADy_n)$$
$$\leq c_2 \max\{d_1(x_{n-1}, Dy_n)d_2(y_n, Bx_{n-1}),$$
$$d_1(x_{n-1}, Dy_n)d_2(y_n, ADy_n), d_2(y_n, Bx_{n-1})d_2(y_n, ADy_n)\}$$
$$d_2^2(y_n, y_{n+1}) \leq c_2 \max\{d_1(x_{n-1}, x_n)d_2(y_n, y_n),$$
$$d_1(x_{n-1}, x_n)d_2(y_n, y_{n+1}), \quad d_2(y_n, y_n)d_2(y_n, y_{n+1})\}$$
$$d_2^2(y_n, y_{n+1}) \leq c_2 \max\{0, d_1(x_{n-1}, x_n)d_2(y_n, y_{n+1}), 0\}$$
$$d_2^2(y_n, y_{n+1}) \leq c_2 d_1(x_{n-1}, x_n)d_2(y_n, y_{n+1})$$
$$\Rightarrow d_2(y_n, y_{n+1}) \leq c_2 d_1(x_{n-1}, x_n)$$

If $d_1(x_n, X_{n+1}) \neq 0$ and by using inequality (2.2), we have,

$$d_2^2(y_n, y_{n+1}) = d_2^2(Bx_{n-1}, ADy_n)$$
$$\leq c_2 \max\{d_1(x_{n-1}, Dy_n)d_2(y_n, Bx_{n-1}),$$
$$d_1(x_{n-1}, Dy_n)d_2(y_n, ADy_n), d_2(y_n, Bx_{n-1})d_2(y_n, ADy_n)\}$$
$$d_2^2(y_n, y_{n+1}) \leq c_2 \max\{d_1(x_{n-1}, x_n)d_2(y_n, y_n),$$
$$d_1(x_{n-1}, x_n)d_2(y_n, y_{n+1}), \quad d_2(y_n, y_n)d_2(y_n, y_{n+1})\}$$
$$d_2^2(y_n, y_{n+1}) \leq c_2 \max\{0, d_1(x_{n-1}, x_n)d_2(y_n, y_{n+1}), 0\}$$
$$d_2^2(y_n, y_{n+1}) \leq c_2 d_1(x_{n-1}, x_n)d_2(y_n, y_{n+1})$$
$$\Rightarrow d_2(y_n, y_{n+1}) \leq c_2 d_1(x_{n-1}, x_n)$$

If $d_2(y_n, y_{n+1}) \neq 0$, it follows that

$$d_1(x_n, x_{n+1}) \leq c_1 d_2(y_n, y_{n+1}) \leq c_1 c_2 d_1(x_{n-1}, x_n) \ldots\ldots \leq (c_1 c_2)^n d_1(x, x_1)$$

and since $0 \leq c_1 \cdot c_2 < 1$, $\{x_n\}$ is a Cauchy sequence with limit $z \in X$ and $\{y_n\}$ is a Cauchy sequence with limit $x \in Y$.

Now suppose that A is a continuous. Then,

$$\lim_{n\to\infty} Ax_n = Az = \lim_{n\to\infty} y_{n+1} = w$$

and so $Az = z = ADw = w$

Now using inequality (4), we get,

$$d_2^2(Bz, y_n) = d_n(Bx_n, ADy_{n-1})$$
$$\leq c_2 \max\{d_1(z, x_{n-1})d_2(y_{n-1}, Bz),$$
$$d_1(z, x_{n-1})d_2(y_{n-1}, y_n), d_2(y_{n-1}, Bz)d_2(y_{n-1}, y_n)\}$$

Letting $n \to \infty$, we have,

$$d_2^2(Bz, w) = c_2 \max\{d_1(z, z)d_2(z, Bz),$$
$$d_1(z, z)d_2(w, w), d_2(w, Bz)d_2(w, w)\}$$
$$d_2^2(Bz, w) \leq 0$$

So, $Bz = w = BCw = w.$

Using inequality (2.1), we get,

$$d_1^2(Cw, x_{n+1}) = d_1^2(Cy_n, DBx_n)$$
$$\leq c_1 \max\{d_2(w, y_{n+1})d_1(x_n, Cw),$$
$$d_2(w, Bx_n)d_1(x_n, Dy_{n+1}), d_1(x_n, Cw)d_1(x_n, x_{n+1})\}$$
$$d_1^2(Cw, x_{n+1}) \leq c_1 \max\{d_2(w, y_{n+1})d_1(x_n, Cw),$$
$$d_2(w, Bz)d_1(x_n, x_{n+1}), d_1(x_n, Cw)d_1(x_n, x_{n+1})\}$$

Letting $n \to \infty$, we get,

$$d_1^2(Cw, z) \leq 0$$
$$\Rightarrow Cw = z = CAz$$

Again using inequality (2.1), we get,

$$d_1^2(z, Dw) = d_1^2(Cw, DBz)$$
$$\leq c_1 \max\{d_2(w, Bz)d_1(z, Cw), d_2(w, Bz)d_1(z, DBz), d_1(z, Cw)d_1(z, DBz)\}$$
$$d_1^2(z, Dw) \leq 0$$
$$\Rightarrow Dw = z = DBz$$

The same result of course add if one of mapping B, C, D is continuous instead of A.

Uniqueness

Suppose that DB has a second fixed point z'. Then by inequality (2.1) and (2.2), we have,

$$d_1^2(z, z') = d_1^2(CAz, DBz')$$
$$\leq c_1 \max\{d_2(Az, Bz')d_1(z', CAz),$$
$$d_2(Az, Bz')d_1(z', DBz'), d_1(z', CAz)d_1(z', DBz')\}$$
$$\leq c_1 \max\{d_2(Az, Bz')d_1(z', z),$$
$$d_2(Az, Bz')d_1(z', z'), d_1(z', z)d_1(z', z')\}$$
$$d_1^2(z, z') \leq c_1 d_2(Az, Bz')d_1(z, z')$$
$$d_1(z, z') \leq c_1 d_2(Az, Bz')$$

By using inequality (2.2), we get,

$$d_2^2(Az, Bz') = d_2^2(ACw, BDBz')$$
$$\leq c_2 \max\{d_1(Cw, DBz')d_2(Bz', ACw),$$
$$d_1(Cw, DBz')d_2(Bz', BDBz'), d_2(Bz', ACw)d_2(Bz', BDBz')\}$$
$$d_2^2(Az, Bz') \leq c_2 \max\{d_1(z, z')d_2(Bz', Az),$$
$$d_1(z, z')d_2(Bz', Bz'), d_2(Bz', Az)d_2(Bz', Bz')\}$$
$$d_2^2(Az, Bz') \leq c_2\{d_1(z, z')d_2(Bz', Az)\}$$
$$d_2(Az, Bz') \leq c_2 d_1(z, z')$$
$$d_1(z, z') \leq c_1 d_2(Az, Bz') \leq c_1 c_2 d_1(z, z')$$

Since $0 \leq c_1 \cdot c_2 < 1$, the uniqueness of z follows.Similarly z is the unique fixed point of CA and w is the unique fixed point of BC and AD. This complete the proof of the theorem.

Corollary 2.1 *Let (X, d_1) be complete metric space.*

Let $A, B, C, D: X \to X$ satisfying the inequality,

$$d_1^2(Cy, DBx) \leq c\max\{d_1(y, Bx)d_1(x, Cy),$$
$$d_1(y, Bx)d_1(x, DBx), d_1(x, Cy)d_1(x, DBx)\}$$
$$d_1^2(Bx, ADy) \leq c\max\{d_1(x, Dy)d_1(y, Bx),$$
$$d_1(x, Dy)d_1(y, ADy), d_1(y, Bx)d_1(y, ADy)\}$$

$\forall\ x, y \in X$ where $0 \leq c < 1$ if one of mappings *A, B, C* and *D* is continuous then *CA* and *DB* have a unique common fixed point *z* and *BC* and *AD* have a unique common fixed point *w*. Further,

$$Az = Bz = w \text{and} Cw = Dw = z.$$

Corollary 2.2 *Let (X,d_1) and (Y, d_2) be complete metric spaces. Let A, B: X→Y and C, D:Y→X satisfying the inequalities,*

$$d_1^2(Cy, DBx) \leq a_1 d_2(y, Bx)d_1(x, Cy) + b_1 d_2(y, Bx)$$
$$d_1(x, DBx) + c_1 d_1(x, Cy)d_1(x, DBx)$$
$$d_2^2(Bx, ADy) \leq a_2 d_1(x, Dy)d_2(y, Bx) + b_2 d_1(x, Dy)$$
$$d_2(y, ADy) + c_2 d_2(y, Bx)d_2(y, ADy)$$

$\forall\ x \in X$ and $y \in Y$ where $a_1, b_2, c_1, a_2, b_2, c_2 \geq 0$ and $(a_1 + b_1 + c_1)\cdot(a_2 + b_2 + c_2) < 1$ then *CA* and *DB* have a unique common fixed point $z \in X$ and *BC* and *AD* have a unique common fixed point $w \in Y$. Further,

$$Az = Bz = w \text{and} Cw = Dw = z.$$

Corollary 2.3 *Let (X, d_1) be complete metric space.*

Let $A, B, C, D: X \mapsto X$ satisfying the inequality,

$$d_1^2(Cy, DBx) \leq a_1 d_1(y, Bx)d_1(x, Cy) + b_1 d_1(y, Bx)$$
$$d_1(x, DBx) + c_1 d_1(x, Cy)d_1(x, DBx)$$
$$d_1^2(Bx, ADy) \leq a_2 d_1(x, Dy)d_1(y, Bx) + b_2 d_1(x, Dy)$$
$$d_1(y, ADy) + c_2 d_1(y, Bx)d_1(y, ADy)$$

$\forall\ x, y \in X$ where $a_1, b_1, c_1, a_2, b_2, c_2 \geq 0$ and $a_1 + b_1 + c_1 < 1$ and $a_2 + b_2 + c_2 < 1$ then *CA* and *DB* have a unique common fixed point *z* and *BC* and *AD* have a unique common fixed point *w*. Further,

$$Az = Bz = w \text{and} Cw = Dw = z.$$

REFERENCES

[1] B. Fisher (1981). Fixed point on two metric spaces, Glasnik Matem. 16, no. 36, pp. 333-337.

[2] Vishal Gupta and Richa Sharma (2012). Fixed point theorems for Mappings, Advances in Applied Science Research 3, no. 3, pp. 1266-1270.

[3] Valeriu Popa (1991). Fixed points on two complete metric spaces, Univ. U

NovomSaduZb. Rad. Prirod{ Mat. Fak. Ser. Mat. 21, pp. 83-93.

[4] B. Fisher and P.P. Murthy, Related xed point theorems for two pairs of map-pings on two metric spaces, Kyung pook Math. J. 37 (1997), 343{347.

[5] R.K. Namdeo and B. Fisher (2003). A related xed point theorem for two pairs of mappings on two complete metric spaces, Non-linear Analysis Forum 8, no. 1, pp. 23-27.

[6] B. Fisher (1982). Related xed points on two metric spaces, Math. Seminar Notes (Kobe University, 10), pp. 17-26.

[7] L. Kikina and K. Kikina (2010). Generalized Fixed point theorems in two metric spaces under implicit relations, Int. J. Math. Analysis 4, No. 30, pp. 1483-1489.

[8] S. C. Nesic (2001). A xed point theorem in two metric spaces, Bull. Math. Soc. Sci. Math. Reumania 44(94), no. 3, pp. 253-257.

[9] R.K. Namdeo, N.K. Tiwari, B. Fisher, and K. Tas (1998). Related xed point theorems on two complete and compact metric spaces, Internat. J. Math. Sci. 21, no. 3, 559-564.

[10] V. Gupta, A. Kanwar, and R.K. Saini (2012). Related Fixed Point Theorem on n Metric Spaces, International Journal of Mathematical Archive 3, no. 3, pp. 902-908.

[11] V. Gupta, A. Kanwar, and S.B. Singh (2012). A Fixed Point Theorem For n-Mapping on n-Metric Spaces, International journal of Emerging Trends in Engineering and Development 1, no. 2, pp. 257-268.

[12] V. Gupta, A. Kanwar, and R. Kaur (2012). A Fixed Point theorem on Five Complete Metric spaces, International journal of Emerging Trends in Engineering and Development 2, no. 2, pp. 202-209.

[13] V. Gupta, N. Mani, and A. Kanwar (2012). A Fixed point Theorem on Four Complete Metric Spaces, International Journal of Applied Physics and Mathematics 2, no. 3, pp. 169-171.

[14] V. Gupta, A.K. Tripathi, and A. Kanwar (2012). Related Fixed Points on Four Metric Spaces, International Journal of Mathematical Sciences 11, no. 3-4, pp. 365-371.

[15] V. Gupta, A. Kanwar, and R. Sharma (2012). Theorem on Fixed Point in n-Metric Spaces, International Journal of Mathematical Archive 3, no. 6, pp. 2393-2399.

[16] V. Gupta (2012). Fixed Point Theorems for Pair of Mappings on Three Metric Spaces, Advances in Applied Science Research, Pelagia Research Library 3, no. 5, pp. 2733-2738.

[17] V. Gupta and A. Kanwar (2013). A Theorem on Fixed Points For Five Metric Spaces Using Rational Inequality, proceeding of national conference, Advancements in the era of Multi - Disciplinary Systems, Elsevier chapter 19, pp. 110-114.

[18] A Related Fixed Point Theorem for n-Metric Spaces using Continuity, Acta Universities Apulensis Seria Mathematics-Informatics 32 (2012), pp. 187-194.

Corresponding Author

Anil Saini*

Assistant Professor of Mathematics, Pt. C. L. S. Govt. College, Sector – 14, Karnal, Haryana