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An Overview on the Properties of Vector-Valued Measurable Functions

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Abstract – This paper intends to study the Vector –Valued Measurable functions and its related properties. Study in this paper is not restricted to, but, we do have extensively dealt with the class of 'Riemann measurable' vector-valued functions and 'Lusin type property'. This class contains all Riemann integrable functions and is closely related to the restricted versions of the McShane and Henstock integrals, the M-and H-integrals, defined by means of Lebesgue measurable gauges. Not exclusively but primarily, in this paper, our developments are in the spirit of the Riemann type integral theory for real-valued functions. In particular, we prove that a bounded Riemann measurable vector-valued function is M -integrable.

Keywords:- Vector-Valued Function, Riemann measurable, Lusian Property, Banach Space, Measurable Space, Riemann Integrable, M-integrable, H-integrable.

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INTRODUCTION

Recalling a "Banach space" we know that there are two basic notions of function measurability. They are the notions of *Bochner* (or *strong*) measurability and *scalar* (or *weak*) measurability. And their relationship is well-known: the Pettis Measurability Theorem states that a function is Bochner measurable if and only if it is both scalarly measurable and almost separably-valued. As a result, these notions of measurability diverge sharply for non-separable range spaces. Two classical examples illustrate some of the difficulties that occur in the non-separable case when dealing with various collections of measurable and integrable vector-valued functions. Although all bounded Bochner measurable functions are necessarily Bochner integrable, according to Graves' example some fairly simple functions exist that is Riemann integrable but not integrable in the Bochner sense. The difficulty is that the function of Graves' example is not the limit of a sequence of finitely-valued Bochner measurable functions. On the other hand, Pettis' theory, which has the widest range among the classical theories of vector-valued integration, does not assign an integral to a bounded scalarly measurable function from Phillips' example.

We originally set out to find a notion of measurability for a vector-valued function that is more relevant to *Riemann type* integration theories, such as those of McShane and Henstock, rather than that of Bochner or scalar measurability. Seeking such a notion of measurability, we turned to the integration theories set forth by Kolmogorov and Birkhoff. These two theories of integration, which are also based on finite or infinite Riemann type sums, turn out to be equivalent and to have all the reasonable generality. They are, however, *not* as simple and as useful as the theory of Riemann type integrals. Later investigations of the Kolmogorov–Birkhoff construction can be found. In connection with some of these investigations several classes of 'measurable' functions were defined that included the collection of Bochner measurable functions as a subclass. These classes consist of functions that are, in a certain sense, very close to Riemann integrable functions and are defined by means of Cauchy type conditions and limit processes.

In this paper we introduce the notion of *Riemann* measurability, generalizing the well-known Lusin condition, which is equivalent to Lebesgue measurability for real-valued functions defined on [*a*, *b*]. The notion of Riemann measurability, which we believe to be new, is based on a weakening of the Lusin condition in which the sets on which the function is required to be continuous are replaced with sets over which the function satisfies a Cauchy type condition for Riemann integrability, so that the function may even be everywhere discontinuous on these sets. Several authors, including Jeffery ('*measurable*' functions), Kunisawa (*-measurable* functions), Snow (P_ε*-measurable* functions or *almost Riemann-integrable* functions), and, more recently, Cascales and Rodríguez (the *Bourgain property*), have used similar notions of measurability in their treatment of the Birkhoff integral. However, we should emphasize that our notion of function measurability, unlike in the papers mentioned above, is formulated without the use of partitions into measurable sets or considering the relation of the function to any special function sequence. Our measurable function class is defined by means of the classical Riemann sums and constant gauges and is therefore closely related to the *M* - and *H* -integrals that are obtained if we assume that the gauge in the definitions of the McShane and Henstock integrals can be chosen to be Lebesgue measurable. Finally, we demonstrate that the class of Riemann measurable functions is large enough to include all Birkhoff integrable functions, while we try to keep, in part at least, the simplicity and usefulness that characterize the theory of Riemann type integrals for real-valued functions defined on a compact interval of the real line.

TERMINOLOGY AND NOTATION USED:

For the most part, our notation and terminology are standard. Throughout this paper [*a, b*] will denote a fixed nondegenerate interval of the real line and *I* (or sometimes *J*) its closed nondegenerate subinterval. *X* denotes a real Banach space and X^* its dual. Let *E* and *H* be sets, then dist(*E, H*) is the *distance* between *E* and *H*; int *E*, ∂*E*, χ_E, and λ(*E*) will denote the *interior* of *E*, the *boundary* of *E*, the *characteristic function* of *E*, and *the Lebesgue measure* of *E*, respectively. For ease of notation, we will drop the adjective Lebesgue and refer to measurable sets, negligible sets, and measurable functions. Finally, a (*measurable*) *gauge* on *E* is any (measurable) positive function defined on a (measurable) set *E*.

Definition

(a) A partial McShane partition of $[a,b]$ is a finite collection $P=\{I_k\}_{K=1}^{k}$ such that $\{I_k\}_{K=1}^{K}$ is a collection of pairwise non-overlapping intervals and $tk \in [a,b]$ for each k. P is subordinate to a gauge δ on [a, b] if $Ik \subset (tk-\delta(tk), tk+\delta(tk))$ for each $k. P$ is said to be a McShane partition of [a, b] provided $\{I_K\}_{K=1}^{k}$ covers [a, b].

We say that a function $f: [a,b] \to X$ is McShane integrable on [a, b], with a McShane integral $\varpi \in X$, if for each positive number ε there is a gauge δ on [a, b] such that

$$\left\|\sum_{k=1}^{K}\{f(t_k)-f(t'_k)\}\cdot\lambda(I_k)\right\|<\varepsilon$$

whenever $\{(I_k,t_k)\}_{k=1}^{K}$ is a McShane partition of [*a, b*] subordinate to *δ*.

A *partial Henstock partition* of [*a, b*] is a partial McShane partition (McShane patition) $\{(I_k,t_k)\}_{k=1}^{K}$ of [*a, b*] with $t_k \in I_k$ for each *k*. A function *f* : [*a, b*] → *X* is *Henstock integrble* on [*a, b*], with *a Henstock integral* $\varpi \in X$, if for each positive number *ε* there is a gauge *δ* on [*a, b*] such that for each Henstock partition $\{(I_k,t_k)\}_{k=1}^{K}$ of [*a, b*] subordinate to *δ*. Customarily, we say that a function f is McShane (Henstock) integrable on a set $E \subset [a,b]$ if the function $\int_E f=\int_a^b f\chi_E$ $f\chi_E$ is McShane (Henstock) integrable on [*a, b*] and Standard arguments show that a McShane (Henstock) integrable on [*a, b*] function is McShane (Henstock) integrable on any subinterval *I* of [*a, b*]. Moreover, a McShane integrable on [*a, b*] function is McShane integrable on any measurable subset of [*a, b*]. Finally, recall that *f* is said to be *scalarly measurable* on a measurable set *E* ⊂ [*a, b*] if for each $x^* \in X^*$ the real-valued function x^*f is measurable on *E*.

DEFINING MEASURABILITY AND INTEGRABILITY:

We begin with the fundamental definition of classes of vector-valued functions.

Definition. Let $f:[a,b]\to X$ and let E be a measurable subset of [*a, b*].

(a) f is said to be *Lusin measurable* on *E* if for each *ε* > 0 there exists a closed set $F \subset E$ with $\lambda(E\setminus F)<\varepsilon$ such that the function $f\,|F$ is continuous.

(b) *f* is said to be *Riemann measurable* on *E* if for each *ε* > 0 there exist a closed set *F* ⊂ *E* with *λ*(*E* \ *F*) <*ε* and a positive number *δ* such that

$$\left\|\sum_{k=1}^{K}\{f(t_k)-f(t'_k)\}\cdot\lambda(I_k)\right\|<\varepsilon$$

whenever $\{I_k\}_{k=1}^{K}$ is a finite collection of pairwise non-overlapping intervals with $\max_{1\leqslant k\leqslant K}\lambda(I_k)<\delta$ and $t_k,t'_k \in I_k\cap F$. Some comments are in order at this point. The Pettis Measurability Theorem shows that 'Lusin measurability' of (a) above implies Bochner measurability. Thus Lusin measurability is equivalent to Bochner measurability. It is our understanding that the 'Riemann measurability' of (b) above is explicitly described here for the first time, although we borrow some essential ideas from some previous studies.

The next theorem summarizes the basic properties of Riemann measurable functions.

Theorem 1 .. *Let* $f : [a,b] \to X, g : [a,b] \to X$ *and let* E *be a measurable subset of* [a, b].

(a) *If f is Riemann measurable on E, then αf is Riemann measurable on E for each αϵ*R.

(b) *If* f *and* g *are Riemann measurable on* E, *then* $f + g$ *is Riemann measurable on E.*

(c) *If* f *is Riemann measurable on* E *and* E_1 *is a measurable subset of E, then f is Riemann measurable on* E_1.

(d) *If* f *is Riemann measurable on* E, *then there exists a sequence* $\{f_n\}_{n=1}^{\infty}$ *of pairwise disjoint closed subsets of E such that the set* $E \setminus \bigcup_{n=1}^{\infty} F_n$ *is negligible and f is bounded on* F_n *for each* n.

(e) *If* f *is Riemann measurable on* E *if and only if* $f\chi E$ *is Riemann measurable on* [a, b].

(f) *If* f *is Lusin measurable on* E, *then f is Riemann measurable on* E.

Proof. The proofs of (a) and (b) are not difficult and we leave them to the reader. Fix $\varepsilon > 0$ in the remainder of this proof.

For (c), let a closed set $F \subset E$ and $\delta > 0$ correspond to $\varepsilon/2$ in the definition of Riemann measurability of *f E*. Evidently $\lambda(E_1 \backslash F) \le \lambda(E \backslash F) < \varepsilon/2$. Choose a closed set $F1 \subset E1 \cap F$ such that $\lambda((E1 \cap F) \setminus F1) < \varepsilon/2$.

This gives us

$$\lambda(E_1 \backslash F) \le \lambda(E \backslash F) + \lambda((E_1 \cap F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

It is now clear that δ and F1 correspond to ε in the definition of Riemann measurability of f on E1.

For (d), combine part (c) with the definition of Riemann measurability.

For (e), suppose first that f is Riemann measurable on E. Let a closed set $F \subset E$ and $\delta > 0$ correspond to ε/2 in the definition of Riemann measurability of f on E. Choose a closed set H ⊂ [a, b] \ E such that $\lambda([a,b] \setminus E) - \lambda(H) < \varepsilon/2$. Define F1 = F ∪ H and $\delta 1 = min(\delta, dist(F,H))$ and note that

$$\lambda([a,b] \backslash F_1) \le \lambda(E \backslash F) + \{\lambda([a,b] \backslash E - \lambda(H)\} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Consequently, δ1 and F1 correspond to ε in the definition of Riemann measurability of $f\chi E$ on [a, b]. (c) and the definition of Riemann measurability of $f\chi E$ on E combine to obtain the converse. For (f), choose a sequence {Fn}∞n=1 of pairwise disjoint closed subsets of E and a sequence $\{\delta n\}\infty n = 1$ of positive numbers such that the set $E \setminus n\infty = 1 F_n$ is negligible.

Let $\{I_k\}_{k=1}^{K}$ be a finitite collection of pairwise non overlapping intervals with $\max_{1 \le k \le K} \lambda(I_k) < \delta$, let $t_k, t_k' \in I_k \cap F$ for each k, and compute

$$\left\|\sum_{k=1}^{K} \{f(t_k) - f(t_k')\} \cdot \lambda(I_k)\right\| < \sum_{n=1}^{N} \sum_{k: t_k, t_k' \in F_n} \|f(t_k) - f(t_k')\| \cdot \lambda(I_k)$$
$$< \sum_{n=1}^{N} \frac{\varepsilon}{2^n(b-a)} \sum_{k: t_k, t_k' \in F_n} \lambda(I_k) < \sum_{n=1}^{N} \frac{\varepsilon}{2^n} < \varepsilon$$

It follows that *δ* and *F* correspond to *ε* in the definition of Riemann measurability of $f \text{ on } E$.

Definition. A function $f: [a,b]\, X$ is said to be *M-integrable* (*H -integrable*) on [*a, b*] if it is McShane (Henstock) integrable on [*a, b*] and for each $\varepsilon > 0$ there exists a *measurable* gauge *δ* on [*a, b*] that corresponds to *ε* in the definition of the McShane (Henstock) integral of $f \text{ on } [a,b]$. The function f is *M* -integrable (*H* -integrable) on a set $E \subset [a,b]$ if $f\chi E$ is *M*-integrable (*H* -integrable) on [*a, b*] and $\int_E f = \int_a^b f_{\chi_E}$.

Remark. Solodov first introduced the *M* -integral for vector-valued functions. He proved that a vector-valued function is *M* -integrable on [*a, b*] if and only if it is integrable on [*a, b*] in the Kolmogorov sense. As we noted in the introduction, the Kolmogorov integral (or the *unconditional* Riemann–Lebesgue integral)is in turn equivalent to the Birkhoff integral.

The standard technique can be applied to show that the *M* - and *H* -integrals have typical properties, including the linearity of the *M*- and *H*-integrals, the relation between *M*-

and *H*-integrations and subintervals, and the Hake Theorem for the *H* -integral.

Theorem 2 . *Let* $f: [a,b] \to X, g: [a,b] \to X$ *and let* $\alpha, \beta \in R$.

(a) *If f and g are M-integrable (H -integrable) on [a, b], then* $\alpha f + \beta g : [a,b] \to X$ *is M-integrable (H -integrable) on [a, b] and*

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

If f is M-integrable (H -integrable) on [a, b] and [c, d] is a nondegenerate subinterval of [a, b], then f is M-integrable (H -integrable) on [c, d].

(b) *Let* $c \in (a,b)$. *If* f *is M-integrable (H -integrable) on [a, c] and [c, b], then* f *is M-integrable (H -integrable) on [a, b] and*

$$\int_a^b f = \int_a^c f + \int_b^c f$$

(c) *If f is H -integrable on [a, c] for each* $c(a,b)$ *for each* $c \in (a,b)$ *and the limits* $\lim_{c \to b-} \int_a^c f$ *exists in X then f is M-integrable on [a,b] and*

$$\int_a^b f = \lim_{c \to b-} \int_a^c f$$

A routine proof can be applied to demonstrate the following more involved property of the M -integral.

Before we illustrate the results of this paper let us understand the basic terms and concepts utilized throughout in this paper.

UNDERSTANDING MEASURABLE SPACES

A measurable space is a set S, together with a nonempty collection, S, of subsets of S, satisfying the following two conditions:

1. For any A, B in the collection S, the set A − B is also in S.

2. For any $A_1, A_2; \cdots \in S, \cup A_i \in S$.

The elements of S are called measurable sets. These two conditions are summarized by saying that the measurable sets are closed under taking finite differences and countable unions.

Think of S as the arena in which all the action (integrals, etc) will take place; and of the measurable sets are those that are "candidates for having a size". In some examples, all the measurable sets will be assigned a "size"; in others, only the smaller measurable sets will be (with the remaining meam surable sets having, effectively "infinite size").

Several properties of measurable sets are immediate from the definition.

1. The empty set ø, is measurable. [Since S is nonempty, there exists some measurable set A. So, $A - A = \emptyset$ is measurable, by condition 1 above.]

2. For A and B any two measurable sets, $A \cap B, A \cup B$, and A − B are all measurable.

It follows immediately, by repeated application of these facts, that the measurable sets are closed under taking any finite numbers of intersections, unions, and differences.

3. For $A1; A2; \cdots$ measurable, their intersection, $\cap$Ai, is also measurable. [First note that we have the following set-theoretic identity: $A1 \cap A2 \cap A3 \cap \cdots = A1 - \{(A1 - A2) \cup (A1 - A3) \cup (A1 - A4) \cup \cdots\}$.

Now, on the right, apply condition 1 above to the set-differences, and condition 2 to the union.] Thus, measurable sets are closed under taking countable intersections and unions.

Here are some examples of measurable spaces.

1. Let S be any set, and let S consist only of the empty set $\emptyset$. This is a (rather boring) measurable space.

2. Let S be any set, and let S consist of all subsets of S. This is a measurable space.

3. Let S be any set, and let S consist of all subsets of S that are countable (or finite). This is a measurable space.

4. Let S be any set, and fix any nonempty collection P of subsets of S. Let S be the collection of subsets of S that result from the following construction. First set $S = P$. Now expand S to include all sets that result by

taking differences and countable unions of sets in S. Next, again expand S to include all sets that result by taking differences and countable unions of sets in (the already expanded) S. Continue in this way, and denote by S the collection that results. Then $(S; S)$ is a measurable space. Thus, you can generate measurable spaces by starting with any set S, and any collection P of subsets of S (i.e., those that you really want to turn out, in the end, to be measurable). By expanding that original collection P, as described above, you can indeed achieve a measurable space in which the chosen sets are indeed measurable.

5. Let $(S; S1)$ be any measurable space, and let $K \subset S$ (not necessarily measurable). Let K1 denote the collection of all subsets of K that are S-measurable. Then $(K, K1)$ is a measurable space. [The two properties for $(K, K1)$ follow immediately from the corresponding properties of (S, S1).] Thus, each subset of a measurable space gives rise to a new measurable space (called a subspace of the original measurable space).

6. Let $(S0, S1)$ and $(S00; S01)$ be measurable spaces, based on disjoint unu derlying sets. Set $S = S0 \cup S00$, and let S consist of all sets $A \subset S$ such that $A \cap S1 \epsilon S1$ and $A \cap S01 \varepsilon S01$. Then $(S, S1)$ is a measurable space.

THE CONCEPTS OF MEASURABLE FUNCTIONS:

Lets begin with some motivation from probability. Let $(\Omega; F; P)$ be a probability space. It is known that random variables should be considered as mappings from $\Omega \text{ to } R$. But is this enough for a rigorous mathematical theory. In practise, in calculating probabilities such as $Prob(X > a)$ where $a \epsilon R$. This means terms of the measure P, there must be

$$Prob\,(X > a) = P(\{\omega \in \Omega; X(\omega) \in (a, \infty)\}) = P(X^{-1}(a, \infty))$$

Now $X - 1(a; \infty) \subseteq \Omega$, however P only makes sense when applied to sets in F. So it can be concluded that $Prob(X > a)$ only makes sense if we impose an additional condition on the mapping X, namely that $X - 1(a; \infty)\ \epsilon\ F$ for all $a \epsilon R$. This property is precisely what is meant by measurability.

In fact let $(S; \Sigma)$ be an arbitrary measurable space. A mapping $f : S \to R$ is said to be measurable if $f - 1(a; \infty)\ \epsilon\ \Sigma$ for all $a \epsilon R$. So in particular, we should define a random variable on a probability space to be a measurable mapping from Ω to R.

Theorem 3 : Let f : S → R be a mapping. The following are equivalent:

$f - 1((a; \infty))\ \epsilon\ \Sigma$ for all $a \epsilon R$.

$f - 1([a; \infty))\ \epsilon\ \Sigma$ for all $a \epsilon R$.

$f - 1((-\infty; a))\ \epsilon\ \Sigma$ for all $a \epsilon R$.

$f - 1((-\infty; a])\ \epsilon\ \Sigma$ for all $a \epsilon R$.

Proof. (i) $\Leftrightarrow$ (iv) as $f - 1(A)c = f - 1(Ac)$ and Σ is closed under taking complements.

(ii) $\Leftrightarrow$ (iii) is proved similarly.

(i) $\Rightarrow$ (ii) uses $[a; \infty) = \cap_{n=1}^{\infty} (a - 1/n, \infty)$ and so on

$$f^{-1}([a, \infty)) = \cap_{n=1}^{\infty} f^{-1}((a - 1/n, \infty))$$

and the result follows since Σ is closed under countable intersections.

(ii) $\Rightarrow$ (i) uses

$$f^{-1}((a, \infty)) = \bigcup_{n=1}^{\infty} f^{-1}([a + 1/n, \infty))$$

and the fact that Σ is closed under countable unions.

It follows that f is measurable if any of (i) to (iv) in Theorem 3 is established for all a ϵ R. Now it can be shown that f is measurable if and only if f−1((a; b)) ϵ Σ for all −∞ ≤ a < b ≤ ∞.

A set O in R is open if for every $x\ \epsilon O$ there is an open interval I containing x for which I ⊆ O.

Proposition. Every open set O in R is a countable union of disjoint open intervals.

Proof. For x ϵ O, let Ix be the largest open interval containing x for which Ix ⊆ O. If x; y ϵ O and x ≠y then either Ix and Iy are disjoint or identical, for if they have a non-empty intersection their union is an open interval containing both x and y and that leads to a contradiction unless they coincide. Clearly $O = \cup_{x \in O} I_v$

Now select a rational number r(x) in every interval Ix and rewrite O as the countable disjoint union over intervals Ix labelled by distinct rationals r(x).

It follows that every open interval in R is an open set. Also we see from Proposition 2 that if O is an open set in R then O ϵ B(R).

Theorem 4 The mapping f : S → R is measurable if and only if f−1(O) ϵ Σ for all open sets O in R.

Proof. Suppose that f−1(O) ϵ Σ for all open sets O in R. Then in particular f−1((a; ∞)) ϵ Σ for all a ϵ R and so f is measurable. Then,

$$f^{-1}(O) = \bigcup_{n=1}^{\infty} f^{-1}((a_n, b_n))$$

If f is measurable, then $f^{-1}((a_n, b_n)) \in \Sigma$ for all n ϵ N and so f-1(O) ϵ Σ since Σ is closed under countable union.

Theorem 5 The mapping f : S → R is measurable if and only if f−1(A) ϵ Σ for all A ϵ B(R).

Proof. Suppose that f is measurable and let A = {E ⊆ R;f−1(E) ϵ Σ}. It is first required to show that A is a σ-algebra. S(i). R ϵ A as S = f−1(R).

S(ii). If E ϵ A then Ec ϵ A since f−1(Ec) = f−1(E)c ϵ Σ.

S(iii) If (An) is a sequence of sets in A then $\cup_{n\in N} A_n \in A$ since $f^{-1}(\cup_{n\in N} A_n) = \cup_{n\in N} f^{-1}(A_n) \in \Sigma$ On the basis of the above studies it have been established that f-1((a,b)) ϵ Σ for all $-\infty \le a \le b \le \infty$ and so A is a σ-algebra of subsets of R that contains all the open intervals. But, according to defination B(R) is the smallest of such σ-algebras. It follows that $B(R) \subseteq A$ and so $f^{-1}(A) \in \Sigma$ for al $A \in B(R)$.

The converse is easy (e.g. just allow A to range over open sets, and use above Theorem).

Theorem 6 leads to the following important extension of the idea of a measurable function: Let $(S1; \Sigma1)$ and $(S2; \Sigma2)$ be measurable spaces. The mapping $f : S1 \to S2$ is measurable if $f - 1(A) \in \Sigma1$ for all $A \in \Sigma2$.

Let $(S; \Sigma; m)$ be a measure space and $f : S \to R$ be a measurable function. It is easy to see that the mapping $mf = m ◦ f - 1$ is a measure on $(R; B(R))$. Indeed $mf(\emptyset) = 0$ is obvious and if (An) is a sequence of disjoint sets in B(R) we have

$$mf\left(\bigcup_{n=1}^{\infty} An\right) = m\left(f^{-1}\left(\bigcup_{n=1}^{\infty} An\right)\right) = m\left(\bigcup_{n=1}^{\infty} f^{-1}(An)\right) = \sum_{n=1}^{\infty} m(f^{-1}(An))$$
$$= \sum_{n=1}^{\infty} mf(An).$$

where for $m \neq n, f^{-1}(An) \cap f^{-1}(Am) = f^{-1}(An \cap Am) = f^{-1}(\emptyset) = 0$

The measure mf is called the pushforward of m by f. In the case of a probability space $(\Omega; F; P)$ and a random variable $X : \Omega \to R$, the pushforward is usually denoted p_X. It is a probability measure on $(R; B(R))$ (total mass 1) and is called the probability law or probability distribution of the random variable X.

SOME EXAMPLES OF MEASURABLE FUNCTIONS:

Frst consider the case where $S = R$ (equipped with its Borel σ-algebra)

and look for classes of measurable functions. In fact, it will prove that {continuous functions on $R\}$ $\subseteq$ {measurable functions on R}.

Proposition. A mapping $f : R \to R$ is continuous if and only if $f - 1(O)$ is open for every open set O in R.

Proof. First suppose that f is continuous. Choose an open set O and let a ϵ f−1(O) so that f(a) ϵ O. Then there exists ϵ > 0 so that (f(a) − ϵ, f(a) + ϵ) ⊆ O. By definition of continuity of f, for such an ϵ there exists δ > 0 so that $x \in (a - \delta, a + \delta) \Rightarrow f(x) \in (f(a)$ − ϵ, f(a) + ϵ). But this tells us that (a – δ, a + δ) ⊆ f−1((f(a) − ϵ, f(a) + ϵ)) ⊆ f−1(O). Since a is arbitrary it can be conluded that f−1(O) is open.Conversely suppose that f−1(O) is open for every open set O in R. Choose a ϵ R and let ϵ > 0. Then since (f(a) − ϵ, f(a) + ϵ) is open so is f−1((f(a) − ϵ, f(a) + ϵ)). Since a ϵ f−1((f(a) − ϵ, f(a) + ϵ)) there exists δ > 0 so that (a – δ, a + δ) ⊆ f−1((f(a) − ϵ, f(a) + ϵ)). From here it can be seen that whenever |x – a| < δ we must have |f(x)−f(a)|< ϵ. But then f is continuous at a and the result follows.

Corollary. Every continuous function on R is measurable.

Proof. Let f : R → R be continuous and O be an arbitrary open set in R. Then (O) is an open set in R, f−1(O) is in B(R). Hence f is measurable. There are many discontinuous functions on R that are also measurable. Lets look at an important class of examples in a wider context. Let (S; Σ) be a general measurable space. Fix A ϵ Σ and define the indicator function 1A : S → R by

$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$I_A^{-1}((c,\infty)) = \emptyset \in \Sigma \text{ if } c \geq 1$$
$$I_A^{-1}((c,\infty)) = A \in \Sigma \text{ if } 0 \leq c < 1$$
$$I_A^{-1}((c,\infty)) = S \in \Sigma \text{ if } c < 0$$

If (S, Σ) = (R, B(R)) or indeed if S is any metric space, then $1A$ is clearly a measurable but discontinuous function.

A particularly interesting example is obtained by taking $(S,\Sigma) = (R,B(R))$ and $A = Q$. Then $1A$ is called Dirichlet's jump function.

As already seen that Q is measurable (it is a countable union of points). As there is a rational number between any pair of irrationals and an irrational number between any pair of rationals, we see that in this case $1A$ is measurable, but discontinuous at every point of R. A measurable function from $(R,\ B(R))$ to $(R,B(R))$ is sometimes called Borel measurable.

MEASURABLE FUNCTIONS ALGEBRAS:

Consider, (S, Σ) is a measurable space. Let f and g be functions from S to R and define for all x ϵ S,

$$(f \bigvee g)(x) = \max(f(x), g(x)), (f \bigwedge g)(x) = \min(f(x), g(x))$$

Proposition. If f and g are measurable then so are

Proof. This follows immediately from the facts that for all c ϵ R,

$$(f\vee g)^{-1}((c,\infty)) = f^{-1}((c,\infty)) \cup g^{-1}((c,\infty)) \text{ and}$$
$$(f \wedge g)^{-1}((c,\infty)) = f^{-1}((c,\infty)) \cap g^{-1}((c,\infty))$$

Let $-f$ be the function $(-f)(x) = -f(x)$ for all $x \in S$. If f is measurable it is easily checked $that\ -f$ also is. Let 0 denote the zero function that maps every element of S to zero, i.e. $0 = 1\emptyset$. Then 0 is measurable since it is the indicator factor of a measurable set.

Define $f_+ = f \vee 0$ and $f_- = -f \vee 0$. so that all x ϵ s.

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}, f_-(x) = \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{if } f(x) > 0 \end{cases}$$

Corollary. If f is measurable then so are $f+$ and $f-$.

Now define the set $\{f > g\} = \{x \in S, f(x) > g(x)\}$.

Proposition. If f and g are measurable then {f > g} ϵ Σ.

Proof Let $\{rn, n \in N\}$ be an enumeration of the rational numbers. Then

$$\{f > g\} = \bigcup_{n\in N} \{f > r_n > g\} = \bigcup_{n\in N} \{f > r_n\} \cap (g < r_n)$$
$$= \bigcup_{n\in N} f^{-1}((r_n,\infty)) \cap g^{-1}((-\infty, r_n)) \in \Sigma$$

Theorem 7 If f and g are measurable then so is f + g.

Proof. By now it is known that a −g is measurable for all a ϵ R. Now (f + g)−1((a, ∞)) = {f + g > a} = {f > a – g} ϵ Σ;

by previous Propositions and this establishes the result.

Use induction to show that if $f1, f2 \ldots\ldots f_n$ are measurable and $c1, c2, \ldots\ldots, cn \in R$ then f is also measurable where $f = c1f1 + c2f2 + \cdots .. + cnfn$. So the set of measurable functions from S to R forms a real vector space. Of particular interest are the simple functions which take the form

$$f = \sum_{i=1}^{n} c_i I A_i \text{ where } A_i \in \Sigma(1 \leq i \leq n)$$

Theorem 8 If f : S → R is measurable and $G : R \to R$ is continuous then $G ∘ f$ is measurable from S to R.

Proof. For all $a \in R$ let $Oa = G - 1((a,\infty))$. Then since G is continuous, O a is an open set in R. Then since for any subset A of S, $(G ∘ f) - 1(A) = f - 1(G - 1(A))$, we have

$$(G \circ f)-1((a,\infty)) = f-1(G-1((a,\infty))) = f-1(Oa) \in \Sigma;$$

The result follows.

Theorem 9 if f and g are measurable so is fg

Proof. Apply above Theorem with $G(x) = x2$ to deduce that $h2$ is measurable whenever h is. But $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$ and the result follows.

MEASURABLE FUNCTIONS LIMITS:

Let (fn) be a bounded sequence of functions from S to R such that the condition supnϵNsupxϵS |fn(x)|<∞. Define infnϵNfn and supnϵNfn by

$$\left(\inf_{n\in N} f_n\right)(x) = \inf_{n\in N} f_n(x) \text{ and } \left(\sup_{n\in N} f_n\right)(x) = \sup_{n\in N} f_n(x)$$

Proposition. If fn is measurable for all nϵN then infnϵNfn and supnϵNfn are both measurable.

Proof. For all c ϵ R,

$$\left\{\inf_{n\in N} f_n > c\right\} = \bigcap_{n\in N}\{f_n > c\} \in \Sigma$$

$$\left\{\sup_{n\in N} f_n > c\right\} = \bigcup_{n\in N}\{f_n > c\} \in \Sigma$$

Define $\liminf_{n\to\infty} f_n$ and $\limsup_{n\to\infty} f_n$ by

$$\left(\liminf_{n\to\infty} f_n\right) = \liminf_{n\to\infty} f_n(x) \text{ and } \left(\limsup_{n\to\infty} f_n\right) = \limsup_{n\to\infty} f_n(x)$$

For all x ϵ S.

UNDERSTANDING SIMPLE FUNCTIONS:

Recall the definition of indicator functions IA where A ϵ Σ. A mapping

f : S → R is said to be simple if it takes the form

$$f = \sum_{i=1}^{n} c_i I A_i$$

where $c1, c2, \dots\dots, cn\epsilon R$ and $A1, A2, \dots, An \epsilon \Sigma$ with $\cup_{i=1}^{n} A_i = S$ and $A_i \cap A_j = \emptyset$ when $i \neq j$. In other words, a simple function is a finite linear combination of indicator functions of non-overlapping sets. It follows from above theorems that every simple function is measurable. It is straightforward to prove that sums and scalar multiples of simple functions are themselves simple, so the set of all simple functions form a vector space.

Recall that a mapping $f : S \to R$ is non-negative if $f(x) \geq 0$ for all $x \epsilon S$, which in short is written as $f \geq 0. f \leq g$ when $g - f \geq 0$. It is easy to see that a simple function is non-negative if and only if $ci \geq 0 \; (1 \leq i \leq n)$.

Theorem 10 : Let $f : S \to R$ be measurable and non-negative. Then there exists a sequence (Sn) of non-negative simple functions on S with $s_n \leq s_{n+1} \leq f$ for all $n \in N$ so that Sn converges pointwise to f as $n \to \infty$. If f is bounde then convergence is uniform.

Proof. This problem needs to be broken in three steps:

Step 1- Construction of (S_n).

Divide the interval $[0,n)$ into $n2n$ subintervals $\{I_j, 1 \leq j \leq n2^n\}$, each of length 1/2n by taking $I_j = \left[\frac{j-1}{2^n}, \frac{j}{2^n}\right)$. Let Ej=f-1(Ij) and Fn=f-1([n,∞)). Then $S = \cup_{j=1}^{n2^n} E_j \cup F_n$. for all x ϵ S

$$S_n(x) = \sum_{j=1}^{n2^n} \left(\frac{j-1}{2^n}\right) 1_{E_j}(x) + n 1_{F_n}(x)$$

Step 2 – Properties of (Sn)

For x ϵ Ej, Sn(x) =(j-1)/ 2n and (j-1)/ 2n ≤ f(x) < (1/2n) and so Sn(x) ≤ f(x). For x ϵ Fn, Sn(x) = n and f(x) ≥ n. So it concludes that Sn(x) ≤ f for all n ϵ N. To show that Sn ≤ Sn+1 fix an arbitrary j and consider Ij =[(j-1)/2n,j/2n). For convenienace write Ij as I such that $I = I_1 \cup I_2$ where I1= [(2j-2)/2n+1, (2j-1)/2n+1) and I2 = [(2j-1)/2n+1, 2j/2n+1). Let E= f-1(I), E1=f-1(I1) and E2=f-1(I2). Then Sn(x) = (j-1)/22 for all x ϵ E and so on for X ϵ E1 and x ϵ E2. It follows that Sn ≤ Sn+1 for all x ϵ E.

Step 3 – Convergence of (Sn)

For any x ϵ S, since f(x) ϵ R there exists n0 ϵ N so that f(x) ≤ n0. Then for each n > n0, f(x) ϵ Ij for some 1≤ j ≤ n2n. from here on the basis of above theorems, the result follows, from which the uniformity of convergence is deduced.

WHAT ARE MEASURES:

Let (S,S1) be a measurable space. A measure on (S, S1) consists of a nonempty subset, M, of S1, together with a mapping $M \xrightarrow{\mu} R^+$ (where R+ denotes the sets of non-negative reals) satisfying the following two conditions.

from applying the above

1. For any A ϵ M and any B ⊂ A, with B ϵ S1, we have B ϵ M.

2. Let A1, A2; · · · ϵ M be disjoint, and set A = A1 ∪ A2 ∪ · · ·. Then: This union A is in M if and only if the sum μ(A1) + μ(A2) + · · · converges; and when these hold that sum is precisely μ(A).

A set A ϵ M is said to have measure; and μ(A) is called the measure of A. Think of the collection M as consisting of those measurable sets that actually are assigned a "size" (i.e., of those size-candidates (in S1) that were successful); and of μ(A) as that size.

Then the first condition above says that all sufficiently small measurable sets are indeed assigned size. The second condition says that the only excuse a measurable set A has for not being assigned a size is that "there is already too much measure inside A", i.e., that A effectively has "infinite measure". The last part of condition 2 says that measure is additive under taking unions of disjoint sets (something we would have wanted and expected to be true).

Several properties of measures are immediate from the definition.

1. The empty set ø is in M, and μ(ø) = 0. [There exists some set A ϵ M. Set B = ø and apply condition 1, to conclude øϵ M. Now apply condition 2 to the sequence (having union A = ø). Since A ϵ M, we have μ(ø) + μ(ø) + · · ·- μ(ø), which implies μ(ø) - 0.]

2. For any A, B ϵ M, A ∩ B, A ∪ B, and A−B are all in M. Furthermore, if A and B are disjoint, then μ(A ∪ B) = μ(A) + μ(B). [The first and third follow immediately from condition 1, since A ∩ B and A−B are both subsets of A. For the second, apply condition 2 to the sequence A – B, B, ø,…….. of disjoint sets, with union A ∪ B. Additivity of the measures also follows from this, since when A and B are disjoint, A − B = A.]

3. For any A, B ϵ M, with B ⊂ A, then μ(B)≤ μ(A). [We have, by the previous item, μ(A) = μ(B) + μ(A − B).] Thus, "the bigger the set, the larger its measure".

4. For any A1, A2; · · · ϵ M, ∩Ai ϵ M. [This is immediate from condition 1 above, since ∩Ai ϵ S1 and ∩Ai ⊂ A1 ϵ M.]

Thus, the sets that have measure (i.e., those that are in M) are closed under finite differences, intersections and unions; as well as under countable intersections. What about countable unions? Let A1, A2; · · · be a sequence of sets in M, not necessarily disjoint. First note that $\cup A_i$ = A can always be written as a union of a collection of disjoint sets in M, namely of A1, A2 − A1, A3 − A2 − A1; · · ·. If the sum of the measures of the sets in this last list converges, then, by condition 2 above, we are guaranteed that A ϵ M. And if the sum doesn't converge, then we are guaranteed that A is not in M. Note incidentally, that convergence of this sum is guaranteed by convergence of the sum μ(A1) + μ(A2) + μ(A3) + · · · (but, without disjointness, this last sum may exceed μ(A)). In short, the sets that have measure are not in general closed under countable unions, but failure occurs only because of excessive measure.

Here are some examples of measures.

1. Let S be any set, let S, the collection of measurable sets, be all subsets of S, let M= S, and, for A ϵ M, let μ(A) = 0. This is a (boring) measure.

2. Let S be any set, S all countable (or finite) subsets of S, M the collection of all finite subsets of S, and, for A ϵ M, let μ(A) be the number of elements in the set A. This is is called counting measure on S. Note that the set S itself could be uncountable.

3. Let S be any set and S the collection of all subsets of S. Fix a nonnegative function $S \xrightarrow{f} R^+$ on S. Now let M consist of all sets A ϵ S such that ΣAf converges. Thus, M includes all the finite subsets of S; and possibly some countably infinite subsets (provided there isn't too much f on the subset); and possibly even some uncountable infinite subsets (provided f vanishes a lot on the subset). For A ϵ M, set μ(A) = ΣAf. This is a measure. For f = 1, it reduces to counting measure.

4. Let (S, S1,M,μ) be any measurable space/measure. Fix any K ϵ S (not necessarily in S). Denote by K the collection of all sets in S that are subsets of K; and by MK the collection of all sets in M that are subsets of K. For A ϵ MK, set μK(A) = μ(A). Then (K, K1, MK, μK) is again a measurable space/measure. [This is an easy check, using for each property, the corresponding property of (S, S1, M, μ).] Thus, any subset of the underlying set S of a space with measure gives rise to another space with measure. This is called, of course, a measure subspace.

5. Let (S0,S1, M0,μ0) and (S00,S01,M00, μ00) be measurable spaces/measures, with S0 and S00 disjoint. Set S = S0 ∪ S00; let S consist of A ⊂ S such that A ∩ S0 ϵ S1 and A

$\cap$ S00 ϵ S01. Let S (resp, M) consist of A ⊂ S such that A $\cap$ S0 ϵ S1 and A $\cap$ S00 ϵ S1 (resp, ϵ M0 and ϵ M00). Finally, for A ϵ M, set μ(A) = μ0(A $\cap$ S0) + μ00(A $\cap$ S00). This is a measurable space/measure. Thus, we may take the "disjoint union" of two measurable spaces/measures.

6. Let (S, S) be a measurable space, and let (M, μ) and (M, μ0) be two measures on this space. [Note that they have the same M.] Define M $\xrightarrow{\mu+\mu^0}$ R+ by: (μ + μ0)(A) = μ(A) + μ0(A). This is a measure, too. And, similarly, for any number a > 0, the mapping $M \xrightarrow{a\mu} R^+$ with action (aμ)(A) = aμ(A) is a measure. Thus, we can add measures, and multiply them by positive constants.

We now obtain two results to the effect that "if a sequence of sets apa proaches (in a suitable sense) another set, then their measures approach the measure of that other set". In short, the measure of a set is "a continuous function of the set".

Theorem 11. Fix a measure space (S, S1, M, μ), let A1 ⊂ A2 ⊂ · · · with Ai ϵM; and set A = $\cup$ Ai. Then: A ϵ M if and only if the sequence μ(Ai) of numbers converges (as i→∞); and when these hold that limit is precisely μ(A).

Proof. Since the Ai are nested, we have the following set-theoretic identities:

A = A1 $\cup$ (A2 − A1) $\cup$ (A3 − A2) $\cup$ . . . ;

Ai = A1 $\cup$ (A2 − A1) $\cup$ (A3 − A2) $\cup$ . . . $\cup$ (Ai – Ai−1):

Note that the sets in the unions on the right are disjoint, and in M. Since the union on the right of Eqn. (2) is finite, we have

μ(Ai) = μ(A1) + μ(A2 − A1) + μ(A3 − A2) + · · · + μ(Ai – Ai−1):

Hence: The μ(Ai) converge if and only if the sum μ(A1) + μ(A2 − A1) + μ(A3 − A2) + · · · converges; which in turn holds if and only if A ϵ M and the definition of a measureg; and that when these hold μ(A) = lim μ(Ai) as per above equations and the definition of a measureg.

Theorem 12. Fix a measure space (S; S; M; μ), let A1 ⊃ A2 ⊃ · · ·, with Ai ϵM; and set A = $\cap$ Ai. Then $A \in M$, and $\lim_{i\to\infty} \mu(A_i) = \mu(A)$.

Proof. The proof is similar to that above (but easier), using the fact that A1 = A $\cup$ (A1 −A2) $\cup$ (A2 −A3) $\cup$ . . ., where the sets on the right are disjoint, and in M.

As a final result on measure spaces, we show that, under certain circumc stances, a ($\mathcal{M}$; μ) that is "not quite a measure" can be made into one by including within $\mathcal{M}$ certain additional sets. Let (S, S1) be a measurable space. Let $\mathcal{M}$ be a nonempty subset of S, and let μ be a mapping , $\mathcal{M} \xrightarrow{\mu} R^+$ Let us suppose that this (M; μ) satisfies the following two conditions:

1. For any $A \in M$ and any B ⊂ A, with $B \in S$, we have $B \in M$.

2. Let A1, A2; · · · $\in M$ be disjoint, and set A = A1 $\cup$ A2 $\cup$ . . ., their union. Then, provided $A \in M$, the sum μ(A1) + μ(A2) + · · · converges, to μ(A).

Thus, this (M; μ) is practically a measure on (S,S). Condition 1 above is identical to condition 1 for a measure; and condition 2 is only somewhat weaker than condition 2 for a measure. All that has been left out, in condition 2, is that portion of condition 2 that states:

Whenever Σμ(Ai) converges, then $A \in M$. That is, this (M; μ) is very nearly a measure, lacking only the requirement that disjoint unions of elements of M, if not too obese measurem wise, are themselves in M.

The present result is that, under the circumstances of the paragraph above, we can recover from that (M; μ) a measure. The idea is to enlarge the original M to include the missing sets. Denote by M^ the collection of all subsets of S that are of the form [Ai,where A1; A2; · · · is a sequence of disjoint sets in M for which Σμ(Ai) converges; and let μ^(A) = Σμ(Ai). Note that every set A in M is automatically in M^ ; with μ^(A) = μ(A). This The present theorem is: This (M^ ; μ^) is a measure.

The first step of the proof is to show that the function μ^ is well-defined. To this end, let A = A1 $\cup$ A2 $\cup$ . . . be in M^ via condition ii) above. Let B1, B2, · · · be a second disjoint collection of elements of M, with the same union: $\cup$Bj = A. We must show that Σμ(Bj) = Σμ(Ai), i.e., that μ^(A), defined via the Bj, is the same as μ^(A) defined via the Ai. To see this, set, for i; j = 1; 2; · · ·, Cij = Ai $\cap$ Bj. Then the Cij are disjoint and in M, and their union is precisely A. But by condition 2 we have Σiμ(Cij) = μ(Bj) and Σjμ(Cij) = μ(Ai). That Σμ(Ai) = Σμ(Bj) follows.

To complete the proof, we must show that (M^ ; μ^) satisfies conditions 1 and 2 for a measure. For condition 1: Let $A \in M_\wedge$: We have A = $\cup$Ai, where the Ai are disjoint and are in M, and are such that Σμ(Ai) converges. Let B ⊂ A, with $B \in S$. We must show that $B \in M_\wedge$. But this follows, since B = $\cup$ (B $\cap$ Ai), where the B $\cap$ Ai are disjoint, are in M, and are such that Σμ(B$\cap$ Ai) converges. We leave condition 2 as an (easy) exercise. Here is an example of an application of this result. Let S = Z+, the set of positive integers, let S consist of all subsets of S, let M consist of all finite subsets of S, and, for A ϵ M, let μ(A) = ΣnϵA(1/2n), where the sum on the right is finite. This (M; μ) satisfies conditions 1 and 2 above. But it is not a measure, for it does not satisfy condition 2 for a measure space. In this case, the M^ constructed above consists of all subsets of S, and, for A ϵM^ , μ^(A) = Σn ϵ A(1/2n), where now the sum on the right is over the (possibly infinite) set A. The measure space (M, μ) here constructed will be recognized as a special case of Example above.

Finally, we remark that, when the original (M, μ) of the previous page happens to be a measure, then M^ = M, and μ^ = μ.

We now turn to what is certainly the most important example of a measure space: Lebesque measure. Let S = R, the set of reals. [The case S = Rn is virtually identical, line-for-line, to this case; but S = R makes writing easier.]

Set I = (a; b), an open interval in R. The idea is that we want this interval to be measurable, with measure its length: μ(I) = b − a. Let's try to turn this idea into a measure space. By condition 2 for a measure space, our collection M will have to include also sets of the form K = I1$\cup$ I2 $\cup$· · ·, a union of disjoint intervals, with measure μ(K) = μ(I1) + μ(I2) + · · · provided the sum on the right converges. And furthermore, by condition 1 for a measure space, M will also have to include differences of intervals, i.e., the half-closed intervals [a, b) and (a, b], with measures again b − a. So, we expand our original M to include these new sets. Next, let us return, with this new, expanded M, to condition 2. By this condition, M must include also countable unions of the half-closed intervals. Returning to condition 1, we find that our M must include differences of these unions. Continue in this way, at each stage expanding the then-current M by including the new sets demanded by conditions 2 and 1. Does this process terminate? That is, do we, eventually, reach a point at which applying conditions 2 and 1 to the then-current M does not result in any further expansion of M? If this did occur, then we would be done. Presumably, we would at that point be able to write down some general form for a set in this final M, as well as a general formula for its measure. We would thus have our measure space. But, unfortunately, it turns out that this process does not terminate: Each passage through condition 2 and condition 1 requires that additional, new sets be included in M. In short, this is not a very good way to construct our measure space. So, let's try a new strategy. Fix any set X ⊂ R. Let I1, I2, · · · be any countable collection of open intervals that covers X [i.e., that are such that X ⊂ $\cup I_i$. Note that we do not require that the Ii be disjoint.] There always exists at least one such collection, e.g., (−1, 1), (−2, 2),· · .

Now set m = Σμ(Ii), the sum of the lengths of the Ii. This m is either a nonnegative number or "∞" (in case the sum fails to converge). We define the outer measure of X, written μ∗(X) to be the greatest lower bound of these m's, taken over all countable collections of open intervals that cover X; so μ∗(X) is either a nonnegative number, or "∞" (in case X is covered by no countable collection of intervals the sum of whose lengths converges). The outer measure of X reflects \how much open-interval is required to cover X", i.e., is a rough measure of the "size" of X. For example, for X already an interval, X = (a; b), we have μ∗(X) = (b − a), its length (an assertion that seems rather obvious, but is in fact a bit tricky to prove). As a second example, let X be the set of rational numbers. Order the rationals in any way, e.g., 3/5, −398/57, 3; · · ·. Now fix any ϵ > 0. Let I1 be the interval of length ϵ centered on the first rational (3/5); I2 the interval of length $\epsilon/2$ centered on the second rational (−398/57); and so on. Then these Ii cover X; and μ(I1) + μ(I2) + · · · = ϵ + $\epsilon/2$ + · · · = 2ϵ. But ϵ > 0 is arbitarary: Thus, there exists a covering of X (the rationals) by open intervals the sum of whose lengths is as close to zero as we wish. We conclude: $\mu^*(X)$= 0. The same holds for any countable (or finite) subset of the reals. The outer measure has the sort of behavior we might expect of a measure. For example: For X ⊂ Y ⊂ R, then $\mu^*(X) \le \mu^*(Y)$ (which follows from the fact that any covering of Y is already a covering of X). For X; Y ⊂ R, μ^* (X$\cup$ Y) ≤ $\mu^*(X)$+ $\mu^*(Y)$ (which follows from the fact that the intervals in a covering of X taken together with the intervals in a covering of Y yields a collection of intervals that covers X$\cup$ Y). Thus, it is tempting to try to construct our measure space using outer measure: Let M consist of all subsets X of S = R with finite outer measure, and set μ(X) = $\mu^*(X)$. But, unfortunately, this does not work, as the following example illustrates. For a and b and two numbers in the interval [0; 1), write a $\sim$ b provided a-b is a rational numer. This is an equivalence trlation. Now suppose, for contradiction, that we had a measure space based on outer measure. By the first two properties above, we would have Σ^{μ^*} (Xr) = μ^* ([0; 1)) = 1, where the sum on the left is over all rationals r ϵ [0; 1).

Thus, the outer measure is somewhat flawed as a representative of the"\size" of a set, in the following sense. Certain sets (such as the X above) are, roughly speaking, so frothy that they cannot be covered efficiently by open intervals, and for these the outer measure is "too large".

This observation is the key to finding our measure space. For X and Y any two subsets of S = R, set d(X; Y) = μ^*(X − Y) + μ^* (Y − X), so d(X; Y) is a nonnegative number (or possibly "1"). Think of d(X, Y) as reflecting the extent to which X and Y differ as sets", i.e., as an effective "distance" between the sets X and Y .

This interpretation is supported by the following properties:

1. We have d(X, Y) = 0 whenever X = Y . [But note, that the converse fails, e.g., with Y consisting of X together with any one number not in X.]

2. For any subsets X,Y, Z of R, we have d(X, Z) ≤ d(X,Y) + d(Y,Z). This follows from the facts that X − Z ⊂ (X − Y) ∪ (Y − Z) and Z − X ⊂ (Z − Y) ∪ (Y − X). That is, d(‘) satisfies the triangle inequality.

3. For any subsets X, X0, Y, Y0 of R, d(X ∪ Y, X0 ∪ Y0) ≤ d(X, X0) + d(Y, Y0), and similarly with "∪" replaced by "∩" or "-"{This follows from the fact that the set-difference of X∪ Y and X0 ∪ Y0 is a subset of (X − X0) ∪ (Y − Y0); and similarly for "∩" and "-"{That is, nearby sets have nearby unions, intersections, and differences", i.e., the set operations are "continuous" as measured by d(,).

4. For any subsets X, Y of R, |μ^* (X)− μ^* (Y)| ≤ d(X, Y). This follows from X∪ (Y − X) = Y and Y ∪ (X − Y) = X. That is, outer measure is a d(;)-continuous function of the set.

As we have remarked, the outer measure is sometimes \too large", and this fact renders it unsuitable as a measure. But the outer measure is suitable for generating an effective distance, d(,), between sets, for in this role its propensity to be "too-large" becomes merely an excess of caution.

We now turn to the key definition. Denote by M the collection of all subsets A of S = R with the following property: Given any ϵ > 0, there exists a K ⊂ R, where K is a finite union of open intervals, such that d(A; K) ≤ ϵ. And, for A ϵM, set μ(A) = μ^* (A). In other words, the elements of M are the sets that can be "approximated" (as measured by d(,)) by finite unions of open intervals. And, similarly, μ(A) is approximated by the sum of the lengths of the intervals in K (as follows from the fact that d(A, K) ≤ ϵ implies |μ^* (A) − μ^* (K)| ≤ ϵ). It follows, in particular, that μ(A) is not "∞".

In the land of measure spaces, the more sets that are measurable the better. Do there exists measures that are better, in this sense, than Lebesque measure? That is, does there exist a measure (M^ ; μ^) on R that is an extene sion of Lebesque measure, in the sense that M^ is a proper superset of M, and μ^ agrees with μ on M? It turns out that there does. Let X denote any non-measurable set of finite outer measure. Let S^ consist of all subsets of R of the form (A∩ X) ∪ (B − X), where A and B are measurable. Thus, for example, choosing A = B we conclude that S^ ⊃ S; and, choosing A ⊃ X and B =ø, we conclude that X ϵ S^ . This collection is closed under differences and countable unions (as follows immediately from the fact that S is). Let M^ ⊂ S consist of those sets of this form with B having finite measure; and, for any such set, set $\hat{\mu}$((A\X) [(B − X)) = μ*(A \ X)+μ(B)− μ*(B \ X). Thus, for example, X ϵ M^ , with μ^(X) = μ*(X); and, for A ϵ M, μ^(A) = μ(A). One checks that this (M^ ; μ^) is indeed a measure space, and that it is indeed an extension of Lebesque measure. Since X ϵ M^ but X ϵ M, this is a proper extension.

For the purpose of convenience and better understanding we here, have defined some of the related terms and basic concepts in illustrative format with help of text and pictures.

Riemann Sum

1. Partition the interval [a,b] into *n* subintervals

$$a = x_0 < x_1 \ldots < x_{n-1} < x_n = b$$

- Call the partition P
- The K^{th} subinterval is
- Largest is called the norm, called
- If all subintervals are of equal length, the norm is called regular.

2. Choose an arbitrary value from each subinterval, call it C_i

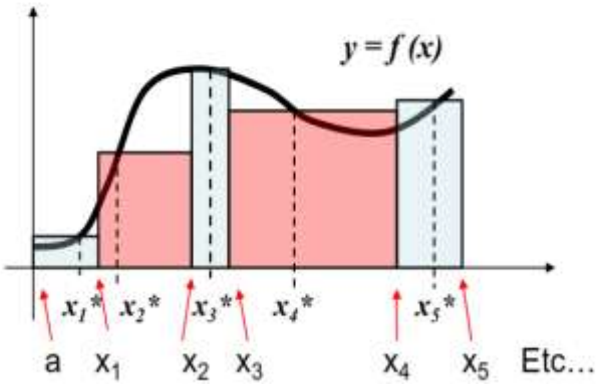
3. Form the sum

$$R_{n} = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + ... + f(c_n)\Delta x_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

This is the Riemann sum associated with

- the function f
- the given partition P
- the chosen subinterval representatives
- We will express a variety of quantities in terms of the Riemann sum

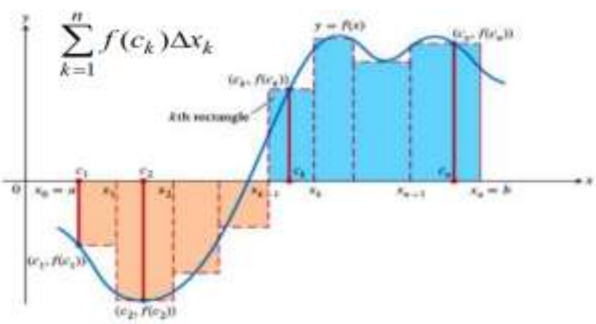
This illustrates that the size of ∆x is allowed to vary



Then a < x1 < x2 < x3 < x4 ….etc. is a partition of [a, b] Notice the partition ∆x does not have to be the same size for each rectangle.

And x1* , x2* , x3* , etc… are x coordinates such that a < x1* < x1, x1 < x2* < x2 , x2 < x3* < x3 , … and are used to construct the height of the rectangles.

The graph of a typical continuous function y = ƒ(x) over [a, b]. Partition [a, b] into n subintervals a < x1 < x2 <…xn < b. Select any number in each subinterval ck. Form the product f(ck)□xk. Then take the sum of these products.



This is called the Riemann Sum of the partition of ∆x.

The width of the largest subinterval of a partition ∆ is the norm of the partition, written ||x||.

As the number of partitions, n, gets larger and larger, the norm gets smaller and smaller.

As n→∞, ||x|| →0 only if ||x|| are the same width!!!!

The Definite Integral

$$I = \int_a^b f(x)dx = \lim_{\|\Delta\| \to 0} \sum_{k=1}^{n} f(c_i)\Delta x_i$$

- The definite integral is the limit of the Riemann sum
- We say that f is integrable when

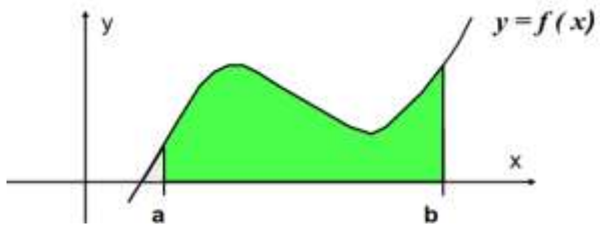
the number I can be approximated as accurate as needed by making || ∆ || sufficiently small

f must exist on [a,b] and the Riemann sum must exist

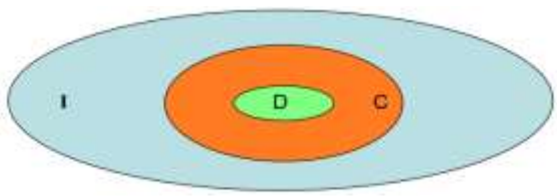
|| ∆ || →0 is the same as saying n→∞

$$\int_a^b f(x)dx$$

The Definite integral above represents the Area of the region under the curve y = f (x), bounded by the x-axis, and the vertical lines x = a, and x = b



Relationship between Differentiability, Continuity, and Inerrability



D – differentiable functions, strongest condition … all Diff'ble functions are continuous and integrable.

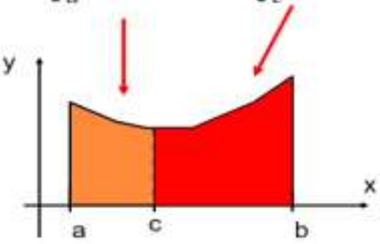
C – Continuous functions, all cont functions are integrable, but not all are diff'ble.

I – integrable functions, weakest condition … it is possible they are not con't, and not diff'ble.

Additive property of integrals

If is integrable over interval [a,b], where a<c<b, then:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



More Properties of Integrals

For *f*, g integrable on [a,b], and *k* is a constant..., then since *kf* and *f*±*g* are integrable on [a,b], we have:

1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

Even – Odd Property of Integrals

For $f(x)$ an even function:

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$$

Even function: $f(x) = f(-x)$ ... symmetric about y - axis

For $f(x)$ an odd function:

$$\int_{-a}^a f(x)dx = 0$$

Inequality Properties

If f is integrable and nonnegative on $[a,b]$:

$0 \leq \int_a^b f(x)dx$

If f, g are integrable on $[a,b]$, and $f(x) \leq g(x)$

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx$$

Now, we here summarize some results about the integration and differentiation of Banach-space valued functions of a single variable. In a rough sense, vector-valued integrals of integrable functions have similar properties, often with similar proofs, to scalar-valued L1-integrals. Nevertheless, the existence of different topologies (such as the weak and strong topologies) in the range space of integrals that take values in an infinite-dimensional Banach space introduces significant new issues that do not arise in the scalar-valued case.

VECTOR-VALUED MEASURABLE FUNCTIONS:

Suppose that X is a real Banach space with norm $\|.\|$ and dual space X' Let $0 < T < \infty$, and consider functions $f: (0,T) \to X$. We will generalize some of the definitions for real-valued functions of a single variable to vector-valued functions.

Measurability: if $E \subset (0,T)$, let

$$\chi_E(t) = \begin{cases} 1 & \text{if } t \in E \\ 0 & \text{if } t \notin E \end{cases}$$

Denote the characteristic function of E

Definition : A simple function $f\ (0,T) \to X$ is a function of the form

$$f = \sum_{j=1}^{N} c_j \chi_{E_j}$$

where E1, . . . , EN are Lebesgue measurable subsets of (0, T) and $c1, \ldots, cN \in X$.

Definition . A function $f : (0,T) \to X$ is strongly measurable, or mea- surable for short, if there is a sequence $\{f_n : n \in N\}$ of simple functions such that $fn(t) \to f(t)$ strongly in X (i.e. in norm) for t a.e. in (0, T).

Measurability is preserved under natural operations on functions.

If $f : (0,T) \to X$ is measurable, then $\|f\| : (0,T) \to R$ is measurable.

If $f : (0,T) \to X$ is measurable and $\varphi : (0,T) \to R$ is measurable, then

$\varphi f : (0,T) \to X$ is measurable.

If $\{f_n : (0,T) \to X\}$ is a sequence of measurable functions and $f_n(t) \to f(t)$ strongly in X for t pointwise a.e. in $(0,T)$, then $f : (0,T) \to X$ is measurable.

We will only use strongly measurable functions, but there are other definitions of measurability. For example, a function $f : (0,T) \to X$ is said to be weakly measurable if the real-valued function $(\omega, f): (0,T) \to R$ is measurable for every $\omega \in X,$. This amounts to a 'co-ordinate wise' definition of measurability, in which we represent a vector-valued function by its real-valued coordinate functions. For finite-dimensional, or separable, Banach spaces these definitions coincide, but for non-separable spaces a weakly measurable function need not be strongly measurable. The relationship between weak and strong measurability is given by the Pettis theorem(1938).

Definition. A function $f: (0,T) \to X$ taking values in a Banach space X is almost separably valued if there is a set $E \subset (0,T)$ of measure zero such that $f((0,T) \setminus E)$ is separable, meaning that it contains a countable dense subset. This definition is equivalent to the condition that $f((0,T) \setminus E)$ is included in a closed, separable subspace of X.

Theorem 13. A function $f: (0,T) \to X$ is strongly measurable if and only if it is weakly measurable and almost separably valued.

Thus, if X is a separable Banach space, $f: (0,T) \to X$ is strongly measurable if and only $(\omega, f): (0,T) \to R$ is measurable for every $\omega \in X,$. This theorem therefore reduces the verification of strong measurability to the verification of measurability of real-valued functions.

Definition. A function $f: [0,T] \to X$ taking values in a Banach space X is weakly continuous if $(\omega, f): [0,T] \to R$ is continuous for every $\omega \in X,$. The space of such weakly continuous functions is denoted by $Cw([0,T]; X)$.

Since a continuous function is measurable, every almost separably valued, weakly continuous function is strongly measurable.

Example. Suppose that H is a non-separable Hilbert space whose dimension is equal to the cardinality of R. Let $\{et : t \in (0,1)\}$ be an orthonormal basis of H, and define a function $f: (0,1) \to H$ by $f(t) = et$. Then f is weakly but not strongly measurable. If $K \subset [0,1]$ is the standard middle thirds Cantor set and {e˜t : t ∈ K} is an orthonormal basis of H, then $g: (0,1) \to H$ defined by $g(t) = 0 \; if \; t \in / K$ and $g(t) = \tilde{e}t \; if \; t \in K$ is almost separably valued since |K| = 0; thus, g is strongly measurable and equivalent to the zero-function.

Example. Define $f: (0,1) \to L\infty(0,1) \; by \; f(t) = \chi(0,t)$. Then f is not almost separably valued, since $\|f(t) - f(s)\|L\infty = 1$ for t $f = s,$ so f is not strongly measurable. On the other hand, if we define g : (0, 1) → L2(0, 1) by g(t) = χ(0,t), then g is strongly measurable. To see this, note that L2(0, 1) is separable and for every $w \in L2(0,1)$, which is isomorphic to L2(0, 1)′, we have

$$(\mathrm{w}, \mathrm{g}(\mathrm{t}))_{L^2} = \int_0^1 w(x) X_{(0,t)}(x) dx = \int_0^1 w(x) dx$$

Thus, $(w, g)L2: (0,1) \to R$ is absolutely continuous and therefore measurable.

Integration. The definition of the Lebesgue integral as a supremum of integrals of simple functions does not extend directly to vector-valued integrals because it uses the ordering properties of R in an essential way. One can use duality to∫define X-valued integrals f dt in terms of the corresponding real-valued integrals $(\omega, f) \, dt$ where $\omega \in X,$', but we will not consider such weak definitions of an integral here. Instead, we define the integral of vector-valued functions by completing the space of simple functions with respect to the $L1(0,T; X) -$norm. The resulting integral is called the Bochner integral, and its properties are similar to those of the Lebesgue integral of integrable real-valued functions.

Definition. Let

$$f = \sum_{j=1}^{N} C_j X_{E_j}$$

Be a simple function and let the integral f be defined by

$$\int_0^T f \, dt = \sum_{j=1}^{N} C_j |E_j| \in X$$

Where |Ej| denotes the Lebesgue measure of Ej.

The value of the integral of a simple function is independent of how it is rep- resented in terms of characteristic functions.

Definition. A strongly measurable function f : (0, T) → X is Bochner integrable, or integrable for short, if there is a sequence of simple functions such that fn(t) → f (t) pointwise a.e. in (0, T) and

$$\lim_{n\to\infty}\int_0^T \|f - f_n\|\, dt = 0$$

The integral of f defined by

$$\int_0^T f\, dt = \lim_{n\to\infty}\int_0^T f_n dt$$

Where the limits exists strongly in X.

The value of the Bochner integral of f is independent of the sequence {fn} of approximating simple functions, and

$$\left\|\int_0^T f\, dt\right\| \le \int_0^T \|f\| dt$$

Moreover, if $A: X \to Y$ is a bounded linear operator between Banach Space X, Y and $f: (0,T) \to X$ is integrable, then $Af: (0,T) \to$ is integrable and

$$A\left(\int_0^T f\, dt\right) = \int_0^T Af\, dt$$

More generally, this equality holds whenever $A: D(A) \subset X \to Y$ is a closed linear operator and $f: (0,T) \to D(A)$, in which case $0\ f\ dt \in D(A)$.

Example. If $f: (0,T) \to X$ is integrable and $\omega \in X$, then $(\omega, f): (0,T) \to R$ is integrable and

$$\left(w, \int_0^T f\, dt\right) = \int_0^T (w, f) dt$$

Example. If J : X→Y is a continuous embedding of a Banach space X into a Banach space Y , and f : (0, T) → X, then

$$J\left(\int_0^T f\, dt\right) = \int_0^T J\, f\, dt$$

Thus the X, Y valued integrals agree, we can identify them.

The following result, due to Bochner (1933), characterizes integrable functions as ones with integrable norm.

Theorem 14. A function $f: (0,T) \to X$ is Bochner integrable if and only if it is strongly measurable and

$$\int_0^T \|f\| dt < \infty$$

Thus, in order to verify that a measurable function f is Bochner integrable one only has to check that the real valued function $\|f\|: (0,T) \to R$, which is necessarily measurable, is integrable.

Example. The functions $f: (0,1) \to H$ and $f: (0,1) \to L\infty(0,1)$ from above examples are not Bochner integrable since they are not strongly measurable. The function $g: (0,1) \to H$ is Bochner integrable, and its integral is equal to zero. The function $g: (0,1) \to L2(0,1)$ is Bochner integrable since it is measurable and $\|g(t)\|_{L^2} = t^{1/2}$ is integrable on (0, 1). The dominated convergence theorem holds for Bochner integrals. The proof is the same as for the scalar-valued case, and we omit it.

Theorem 15. Suppose that $f_n: (0,T) \to X$ is Bochner integrable for each $n \in N, fn(t) \to f(t)\ as\ n \to \infty$ strongly in X for t a.e. in $(0,T)$, and there is an integrable function $g: (0,T) \to R$ such that $\|f_n(t)\| \le g(t)$ for t a.e. in $(0,T)$ and every $n \in N$.

Then $f: (0,T) \to X$ is Bochner integrable and

$$\int_0^T f_n dt \to \int_0^T f\, dt. \quad \int_0^T \|f_n - f\| dt \to 0 \quad as\ n \to \infty$$

As usual, we regard functions that are equal pointwise a.e. as equivalent, and identify a function that is equivalent to a continuous function with its continuous representative.

Theorem 16. If X is a Banach space and $1 \le p \le \infty$, then $Lp(0,T; X)$ is a Banach space.

Simple functions of the form

$$f(t) = \sum_{i=1}^{n} c_i XE_i(t),$$

where $ci \in X$ and E_i is a measurable subset of (0, T), are dense in $Lp(0,T; X)$. By mollifying these functions with respect to t, we get the following density result.

Proposition. If X is a Banach space and 1 ≤ p < ∞, then the collection of functions of the form

$$f(t) = \sum_{i=1}^{n} c_i \phi_i(t)$$

where $\phi_i \in C_c^{\infty}(0.T)$ and $C_i \in X$ is dense $L^p(0,T;X)$.

The characterization of the dual space of a vector-valued Lp-space is analogous to the scalar-valued case, after we take account of duality in the range space X.

Theorem 17. Suppose that 1 ≤ p < ∞ and X is a reflexive Banach space

with dual space X'. Then the dual of $L^p(0,T;X)$ is isomorphic to

$$L^{p'}(0,T;X')$$

where

$$\frac{1}{p} + \frac{1}{p'} = 1$$

The action of $f \in L^{p'}(0,T;X')$ on $u \in Lp(0,T;X)$ is given by

$$\langle\langle f,u\rangle\rangle = \int_o^T \langle f(t),u(t)\rangle\, dt,$$

where the double brackets denote the $L^{p'}(X')$ - $L^p(X)$ duality pairing and the single brackets denote the X' - X duality pairing.

The proof is more complicated than in the scalar case and some condition on X is required. Reflexivity is sufficient (as is the condition that X' is separable).

Differentiability. The definition of continuity and pointwise differ- entiability of vector-valued functions are the same as in the scalar case. A function $f : (0,T) \to X$ is strongly continuous at $t \in (0,T)$ $if\ f(s) \to f(t)$ strongly in $X\ as\ s \to t$, and f is strongly continuous in (0, T) if it is strongly continuous at every point of (0, T). A function f is strongly differentiable at $t \in (0,T)$, with strong pointwise derivative ft(t), if

$$f_t(t) = \lim_{h\to 0}\left[\frac{f(t+h)-f(t)}{h}\right]$$

where the limit exists strongly in X, and f is continuously differentiable in $(0,T)$ if its pointwise derivative exists for every $t \in (0,T)$ and $ft : (0,T) \to X$ is a strongly continuously function. The assumption of continuous differentiability is often too strong to be useful, so we need a weaker notion of the differentiability of a vector-valued function. As for real-valued functions, such as the step function or the Cantor function, the requirement that the strong pointwise derivative exists a.e. in $(0,T)$ does not lead to an effective theory. Instead we use the notion of a distributional or weak derivative, which is a natural generalization of the definition for real-valued functions. Let $L^1_{loc}(0,T;X)$ denote the space of measurable function $f : (0,T) \to X$ that are integrable on every compactly supported interval (a, b) (0, T). Also, as usual, let $C\infty(0,T)$ denote the space of smooth, real-valued functions φ : (0, T) → R with compact support, $supp\ \varphi \Subset (0,T)$.

Definition. A function $f \in L^1_{loc}(0,T;X)$ is differentiable with weak derivate $f_t = g \in L^1_{loc}(0,T;X)$ if $\int_0^T \emptyset' f\,dt = -\int_0^T \emptyset g\,dt$ for every $\emptyset \in C_c^{\infty}(0,T)$ The above integrals are understood as Bochner integrals. In the commonly occurring case where $J : X \hat{\subset} \to Y$ is a continuous embedding $f \in L^1_{loc}(0,T;X$ and $(Jf)_t \in L^1_{loc}(0,T;X)$, from the above example we have

$$J\left(\int_0^T \emptyset' f\, dt\right) = \int_0^T \emptyset' Jf\, dt = -\int_0^T \emptyset (Jf)_t\, dt$$

Thus, we can identify f with Jf and use (6.40) to define the Y -valued derivative of an X-valued function. We then write, for example, that f ∈ Lp(0, T ; X) and ft ∈ Lq(0, T ; Y) if f (t) is Lp in t with values in X and its weak derivative ft(t) is Lq in t with values in Y .

If $f : (0,T) \to R$ is a scalar-valued, integrable function, then the Lebesgue differentiation theorem, implies that the limit $\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f(s)\,ds$ exists and is equal to f(t) for t pointwise a.e. in (0,T). the same result is true for vector-valued integrals. Thus, we can identify f with Jf and use above equation to define the Y -valued derivative of an X-valued function. We then write, for example, that $f \in Lp(0,T;X)$ and $ft \in Lq(0,T;Y)$ if $f(t)$ is Lp in t with values in

X and its weak derivative $ft(t)$ is Lq in t with values in Y.

The Radon-Nikodym property. Although we do not use this dis- cussion elsewhere, it is interesting to consider the relationship between weak differ- entiability and absolute continuity in the vector-valued case. The definition of absolute continuity of vector-valued functions is a natural generalization of the real-valued definition. We say that f : [0, T] → X is absolutely continuous if for every ϱ > 0 there exists a δ > 0 such that

$$\sum_{n=1}^{N} \|f(t_n) - f(t_{n-1})\| < \epsilon$$

for every collection {[t0, t1], [t2, t3], . . . , [tN−1, tN]} of non-overlapping subintervals of [0, T] such that

$$\sum_{n=1}^{N} |t_n - t_{n-1}| < \delta$$

similarly, f : [0, T] → X is Lipschitz continuous on [0, T] if there exists a constant M ≥ 0 such that

$$\|f(s) - f(t)\| \leq M|s - t| \; for\ all\ s,t \in [0,T]$$

It follows immediately that a Lipschitz continuous function is absolutely continuous (with δ = ϱ/M). A real-valued function is weakly differentiable with integrable derivative if and only if it is absolutely continuous c.f. Theorem. This is one of the few properties of real-valued integrals that does not carry over to Bochner integrals in arbitrary Banach spaces. It follows from the integral representation in above Theorem that every weakly differentiable function with integrable derivative is absolutely continuous, but it can happen that an absolutely continuous vector-valued function is not weakly differentiable.

Example. Define $f : (0,1) \to L^1(0,1)$ by $f(t) = t\chi_{[0,t]}$.

Then f is Lipschitz continuous, and therefore absolutely continuous. Nevertheless, the derivative f ′(t) does not exist for any t ∈ (0, 1) since the limit as h → 0 of the difference quotient $\frac{f(t+h)-f(t)}{h}$ does not converge in L1(0, 1), so by above Theorem f is not weakly differentiable.

A Banach space for which every absolutely continuous function has an inte- grable weak derivative is said to have the Radon-Nikodym property. Any reflexive Banach space has this property but, as the previous example shows, the space L1(0, 1) does not. One can use the Radon-Nikodym property to study the geomet- ric structure of Banach spaces, but this question is not relevant for our purposes. Most of the spaces we use are reflexive, and even if they are not, we do not need an explicit characterization of the weakly differentiable functions.

HILBERT TRIPLES: Hilbert triples provide a useful framework for the study of weak and variational solutions of PDEs. We consider real Hilbert spaces for simplicity. For complex Hilbert spaces, one has to replace duals by antiduals, as appropriate.

Definition. A Hilbert triple consists of three separable Hilbert spaces

$$V \hookrightarrow H \hookrightarrow V'$$

such that V is densely embedded in H, H is densely embedded in V′, and

(f, v) = (f, v)H for every f ∈ H and v ∈ V .

Hilbert triples are also referred to as Gelfand triples, variational triples, or rigged Hilbert spaces. In this definition, (·, ·) : V′ × V → R denotes the duality pairing between V′ and V, and (. , .)H : H × H → R denotes the inner product on H.Thus, we identify: (a) the space V with a dense subspace of H through the embedding; (b) the dual of the ‘pivot’ space H with itself through its own inner product, as usual for a Hilbert space; (c) the space H with a subspace of the dual space V′, where H acts on V through the H-inner product, not the V-inner product. In the elliptic and parabolic problems considered above involving a uniformly elliptic, second order operator, we have

$$V = H_0^1(\Omega),\ H = L^2(\Omega), V' = H^{-1}(\Omega), (f,g)_H = \int_\Omega fg\,dx, (f,g)_V = \int_\Omega Df\,.Dg\,dx$$

where $\Omega \subset R^n$ is a bounded open set. Nothing will be lost by thinking about this case. The embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is inclusion. The embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$ is defined by the identification of an L2-function with its corresponding regular distribution, and the action of $f \in L^2(\Omega)$ on a test of functions $v \in H_0^1(\Omega)$ is given by

$$(f,V) \int_\Omega fv dx$$

The isomorphism between $\mathcal{V}$ and its dual space $\mathcal{V}$ is then given by

$$-\Delta: H_0^1(\Omega) \to H^{-1}(\Omega)$$

Thus, a Hilbert triple allows us to represent a ‘concrete’ operator, such as −∆, as an isomorphism between a Hilbert space and its dual. As suggested by this example, in studying evolution equations such as the heat equation ut = ∆u, we are interested in functions u that take values in $\mathcal{V}$ whose weak time-derivatives ut takes values in $\mathcal{V}'$. The basic facts about such functions are given in the next theorem, which states roughly that the natural identities for time derivatives hold provided that the duality pairings they involve make sense.

Theorem 18. Let $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$ be a Hilbert triple. If $u \in L^2(0,T;\mathcal{V})$ and $u_t \in L^2(0,T;\mathcal{V}')$, then $u \in C([0,T];\mathcal{H})$ Moreover:

for any v $\in \mathcal{V}$, the real-valued function $t \mapsto (u(t),v)_{\mathcal{H}}$ is weakly differentiable in $(0,T)$ and

$$\frac{d}{dt}(u(t),v)_{\mathcal{H}} = \langle u_t(t),v\rangle$$

The real valued function $t \mapsto \|u(t)\|^2_{\mathcal{H}}$ is wealkly differentiable in (0,T)

$$And, \qquad \frac{d}{dt}\|u\|^2_{\mathcal{H}} = 2(u_t u);$$

there is a constant C = C(T) such that

$$\|u\|_{L^\infty(0,T;\mathcal{H})} \le C\left(\|u\|_{L^2(0,T;\mathcal{V})} + \|u_t\|_{L^2(0,T;\mathcal{V}')}\right)$$

Proof. We extend u to a compactly supported map $u\tilde{} : (-\infty,\infty) \to V$ with $u\tilde{}t \in L2(R;V')$. For example, we can do this by reflection of u in the endpoints of the interval [4]: Write $u = \varphi u + \psi u \; on \; [0,T]$ where $\varphi,\psi \in C\infty(R)$ are nonnegative test functions such that $\varphi + \psi = 1$ on $[0,T]$ and supp $\varphi \subset [-T/4, 3T/4]$, $supp\; \varphi \subset [T/4, 5T/4]$; then extend φu, ψu to compactly supported, weakly differentiable functions $v,w : (-\infty,\infty) \to V$ defined by

$$v(t) = \begin{cases} \phi(t)u(t) & \text{if } 0 \le t \le T \\ \phi(-t)u(-t) & \text{if } -T \le t < 0 \\ 0 & \text{if } |t| > T \end{cases}$$

$$w(t) = \begin{cases} \psi(t)u(t) & \text{if } 0 \le t \le T \\ \psi(2T-t)u(2T-t) & \text{if } T < t \le 2T \\ 0 & \text{if } |t-T| > T \end{cases}$$

and finally define $u\tilde{} = v + w$. Next, we mollify the extension u˜ with the standard mollifier $\eta\varrho : R \to R$ to obtain a smooth approximation

$$u^\epsilon = \eta^\epsilon * \tilde{u} \in C_c^\infty(R;V), \qquad u^\epsilon(t) = \int_{-\infty}^{\infty} \eta^\epsilon(t-s)\tilde{u}(s)ds$$

The same results that apply to mollifiers of real-valued functions apply to these vector-valued functions. Moreover,as a consequence of the boundedness of the extension operator and the fact that mollification does not increase the norm of a function, there exists a constant 0 < C < 1 such that for all 0 <ε ≤ 1, say,

$$C\|u^\epsilon\|_{L^2(r;V)} \le \|u\|_{L^2(0,T;V)} \le \|u^\epsilon\|_{L^2(r;V)}$$

which implies that $\|u\|^2_{H} : [0,T] \to R$ is absolutely continuous and and above assumptions holds.

Finally, if $\varphi \in C\infty(0,T)$ is a test function $\varphi : (0,T) \to R$ and $v \in V$, then $\varphi v \in C\infty(0,T;V)$. Therefore, since $u^\epsilon_t \to$ ut in $L2(0,T;V')$,

$$\int_0^T \langle u^\epsilon_t, \phi v\rangle dt \to \int_0^T \langle u_t, \phi v\rangle dt$$

Also, since uϱ is a smooth V-valued function

$$\int_0^T \langle u^\epsilon_t, \phi v\rangle dt = -\int_0^T \phi'\langle u^\epsilon, v\rangle dt \to -\int_0^{T^T} \phi'\langle u, v\rangle dt$$

We conclude that for every $\varphi \in C\infty(0,T)$ and $v \in V$

$$\int_0^T \phi\langle u_t, v\rangle dt = -\int_0^T \phi_t\langle u, v\rangle dt$$

We further have the following integration by parts formula.

Suppose that $u,v \in L2(0,T;V)$ and $ut, vt \in L2(0,T;V')$. Then

$$\int_0^T \langle u_t, v\rangle dt = (u(T),v(T))_{\mathcal{H}} - (u(0),v(0))_{\mathcal{H}} - \int_0^T \langle u, v_t\rangle dt$$

Proof. This result holds for smooth functions $u,v \in C\infty([0,T];V)$. Therefore by density and Theorem 6.41 it holds for all functions $u,v \in L2(0,T;V)$ with $ut, vt \in L2(0,T;V')$.

THE RESULTS:

In connection with the exploration of Riemann measurability we will prove the following theorem, which is significant for our analysis.

Theorem 18. Let $E \subset [a,b]$ be measurable. If $f: [a,b] \to X$ is H -integrable on E, then f is Riemann measurable on E.

Proof. We will first prove the theorem in the case in which E = [a, b]. Fix ε > 0. Let a measurable gauge δ0 on [a, b] correspond to $\varepsilon/3$ in the definition of the Henstock integral of f on [a, b]. Since δ0 is measurable on [a, b], there exist δ > 0 and an open set tt such that $\lambda(G) < \varepsilon$ and $\{t \in [a,b]: \delta 0(t) < \delta\} \subset G$. Define F = [a, b] \ G. Let $(I_k)_{k=1}^{K}$ finite collection of pairwise non-overlapping intervals with $\max_{1 \le k \le K} \lambda(I_k) < \delta$ and let $t_k, t_k' \in I_k \cap F$ for each k. As per Saks–Henstock we get

$$\left\| \sum_{k=1}^{K} \{f(t_k) - f(t_k')\} \cdot \lambda(I_k) \right\| \le \left\| \sum_{k=1}^{K} \left\{ f(t_k)\lambda(I_k) - \int_{I_k} f \right\} \right\| + \left\| \sum_{k=1}^{K} \left\{ \int_{I_k} f - f(t_k')\lambda(I_k) \right\} \right\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$$

Now suppose that f is H -integrable on a measurable set E [a, b]. Then the function fχE is Riemann measurable on [a, b]. the function f is Riemann measurable on E. The proof is complete.

The next goal is to prove that any bounded Riemann measurable vector-valued function is M -integrable. Two intermediate results are required before this can be proved. Given a real-valued function f defined on $[a,b]$, recall that $\omega(f,E) = sup\{|f(s) - f(t)|: s,t \in E\}$ is the oscillation of the function f on a set $E \subset [a,b]$.

Theorem 19. Let $f: [a,b] \to X$ and let $E \subset [a,b]$ be measurable. If f is both bounded and Riemann measurable on E, then f is M -integrable on E.

Proof. Without loss of generality, we may assume that E = [a, b]. Suppose that f is both bounded and Riemann measurable on [a, b]. Set $M = \sup_{t \in [a,b]} \|f(t)\| > 0$. Fix ε > 0. Let a closed set $F \subset [a,b]$ and $\delta > 0$ correspond to $min(\varepsilon/20, \varepsilon/4M)$ in the definition of Riemann measurability of f on $[a,b]$. Define a measurable gauge δ0 on [a, b] by

$$\delta_0(t) = \begin{cases} \delta/20 \text{ ift} \in F, \\ dist(t,F), \text{ if } t \in [a,b] \backslash F. \end{cases}$$

Choose two McShane partition of [a,b], $p' = \{(I_k', t_k')\}_{k=1}^{K}$ and $p'' = \{(I_l'', t_l'')\}_{l=1}^{L}$, Which are subordinate to δ_0. Let $S = \{(k,l): 1 \le k \le K, 1 \le l \le L$ then let,

$$S = \{(k,l): 1 \le k \le K, 1 \le l \le L \text{ then let,}$$
$$S_0 = \{(k,l) \in S: t_k' \in F \text{ and } t_l'' \in F\}$$

The non-degenerate intervals of the collection $\{I_k' \cap I_l'': (k,l) \in S$ are pairwise non-overlapping and cover [a, b]. Note that

$$\Delta_f(\mathcal{P}', \mathcal{P}'') = \left\| \sum_{k=1}^{K} f(t_k')\lambda(I_k') - \sum_{l=1}^{L} f(t_l'')\lambda(I_l'') \right\|$$
$$= \left\| \sum_{(k,l)\in S} \{f(t_k') - f(t_l'')\} \cdot \lambda(I_k' \cap I_l'') \right\| \le \| {}_{(k,l)\in S_0} \{f(t_k') - f(t_l'')\} \cdot \lambda(I_k' \cap I_l'') \|$$
$$+ \sum_{(k,l)\in S \backslash S_0} \|f(t_k') - f(t_l'')\| \cdot \lambda(I_k' \cap I_l'') = T_1 + T_2$$

with obvious notation for the terms T1 and T2.

Now we need to estimate these two terms. Since F and δ correspond to min(ε/20, ε/4M) in the definition of Riemann measurability of f on [a, b], we have λ([a, b] \ F) < ε/4M and

$$\left\| \sum_{n=1}^{N} \{f(\tau_n) - f(\tau_n')\} \cdot \lambda(J_n) \right\| < \frac{\varepsilon}{20}$$

By the construction of δ0, we obtain

$$T_2 \le 2M \cdot \sum_{(k,l)\in S \backslash S_0} \lambda(I_k' \cap I_l'') < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}$$

$$\max_{k: t_k' \in F} \lambda(I_k') < \frac{\delta}{10}, \text{ and } \max_{l: t_l'' \in F} \lambda(I_l'') < \frac{\delta}{10}$$

Naturally, the last two inequalities imply that $\max_{(k,l)\in S_0} \lambda(I_k' \cap I_l'') < \delta/10 < \delta/5$. It follows that $\Delta f(P', P'') < \varepsilon$. So, f satisfies the Cauchy criterion for M -integrability on $[a,b]$. This completes the proof.

Corollary. Let $f: [a,b] \to X$ and let $E \subset [a,b]$ be measurable. If the set $f(E \backslash N)$ is separable for some negligible set N ⊂ E and f is Riemann measurable on E, then f is Lusin measurable on E.

Proof. Since f is Riemann measurable on E, f must be scalarly measurable on E. Now the Pettis Measurability Theorem applies to f to show that f is Bochner measurable on E. Since Bochner measurability and Lusin measurability are equivalent, the corollary follows.

We close this paper with a few comments on Riemann measurable functions and on the M - and H

-integrals. Unfortunately, we have been unable to arrive at any idea as to how wide the Riemann mea- surable function class for an arbitrary range space may be. It would be interesting to find some classes of non-separable Banach spaces in which McShane (or even Pettis) integrability implies Riemann measur- ability. This case does not have obvious solutions, but some guidance may be derived from Fremlin and Mendoza give a bounded l∞-valued (note that l∞ is non-separable, but the unit ball of (l∞)* is w*-separable) function that is Talagrand integrable, but not McShane integrable on [0, 1].Their function must be Pettis integrable, but not Riemann measurable on [0, 1]. On the other hand, Fremlin shows that the Birkhoff integral and the McShane integral are still equivalent when the range space has w*-separable dual unit ball (equivalently, when it is linearly isometric to a subspace of l∞). Note that the latter condition is, of course, fulfilled for separable range spaces. Combining Fremlin's result, and above theorem makes it plain that, when the range space is within the above class, a McShane integrable function is nec- essarily Riemann measurable. Fremlin's proof, however, uses the notion of unconditional convergence of an infinite series of elements in a Banach space. Solodov demonstrates that the Kolmogorov integral (and in fact the Birkhoff integral, is equivalent to the M -integral. For this reason, it is also unclear whether a result analogous to Fremlin's above could be valid for the pair of the non-absolute H - and Henstock integrals.

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