To Study on Matrices and Vector Spaces of Graphs

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Abstract – We are presenting a research in mathematics on the linear algebra and matrix. Mathematical model is the area of mathematics which focuses on vector analysis, abstract algebra (also referred to as linear space), linear maps (also referred to as linear transformations) and linear function structures. Vector spaces are a key concept of contemporary mathematics; therefore, linear algebra is commonly used in both abstract algebra and functional analysis. Linear algebra also has a concrete expression in analytic geometry and it is extended in function theory. It has broad applications in the natural sciences and the social sciences, because nonlinear models can sometimes be approximated by linear ones.

Keywords: Matrices, Vector Spaces, Graphs;

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INTRODUCTION

Graphs are the primary subjects of discrete mathematics analysis. They are generally defined as a collection of vertices, V, linked by a set of boundaries, E, each of which is a pair of vertices. Graphs encode ties and are one of the data representations most widely used. Mathematicians also abstractly describe graphs. For eg, we define a path graph as a vertex graph $V = \{1, \ldots, n\}$ and $E = \{(i, i+1) : 1 \leq i < n\}$. Or a number theory might consider a set of pairs for which I divide j = V = {1, ..., n}, and E. The public learns all about social network graphs where and individual is a summit and there are edges between couples who are "mates." Chemists look at graphs that bind the atoms to a molecule. Physicists consider diagrams illustrating molecular interactions. The graphs of one discipline which have nothing in common with that of another discipline.

MATRIX REPRESENTATION OF GRAPHS

The adjacent graph matrix $G = (V, E)$ is a matrix n × n matrix $\mathbf{D} = (d_{ij})$, where n is the sum of vertices in $G, V = \{v_1, \ldots, v_n\}$ and d_{ij} = number of edges from v_i to v_j.
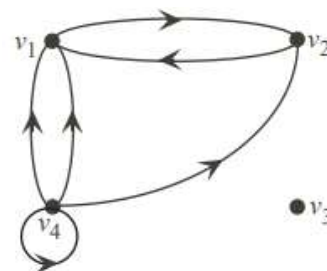
Specifically, $d_{ij} = 0$ if (v_i, v_j) is not a *G* edge. The matrix *D* is symmetrical, that is to say. *DT = D.*
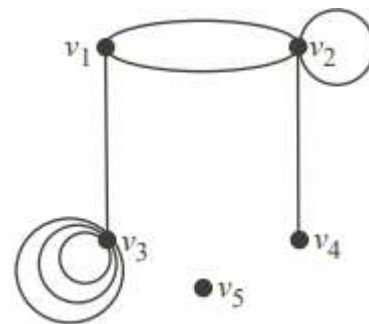
$$\mathbf{D} = \begin{pmatrix} 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An adjacent matrix clearly defines a graph entirely up to an isomorphism. The adjacent matrix of graph *G* is $D = (d_{ij})$, in which case

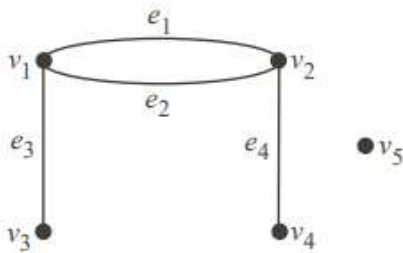d_{ij} = number of arcs coming out of vertex *vi* into vertex *vj*

$$D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

The non-vaulting and loop less Graph $G = (V, E)$ matrix of all-vertex incidences is a $n = m$ matrix $A = (a_{ij})$ where n represents the number of vertices in G, m is the number of borders in G and

$$a_{ij} = \begin{cases} 1 \text{ if } v_i \text{ is an end vertex of } e_j \\ 0 \text{ otherwise.} \end{cases}$$
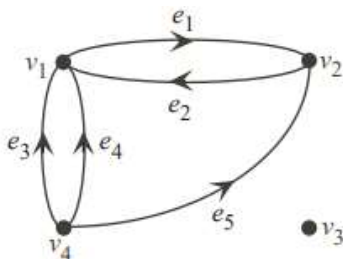
Example:

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \end{array} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \\ \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \end{array}$$

The non-empty and looplessly oriented graph G all-vertex incidence matrix is $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} 1 \text{ if } v_i \text{ is the initial vertex of } e_j \\ -1 \text{ if } v_i \text{ is the terminal vertex of } e_j \\ 0 \text{ otherwise.} \end{cases}$$

Example:

$$A = \begin{array}{cccccc} & e_1 & e_2 & e_3 & e_4 & e_5 \\ & \begin{pmatrix} 1 & -1 & -1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} & \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \end{array}$$

Since each column of an all-vertex matrix contains precisely two non-null numbers, combine multiple, we can remove a row and still have enough information to define the graph. By removing a row from the all-vertex incidence matrix, you obtain the incidence matrix of a graph. It is not unique because n rows can be removed. The vertex adjacent to the removed row is considered the reference vertex.

Likewise, each column of two non-null numbers +1 and −1 on a digraph's all-vertex incidence matrix. A line from the all-vertex incidence matrix can be removed and the incidence matrix can be achieved. Note that the rows of an all-vertex matrix depend linearly because the rows sum is zero.

Theorem 1. Whether a tree is a directed graph or not, the determinant of a nontrivial tree incidence matrix is ±1.

Proof. Proof. The number of vertices in the tree is used induction on n.

Basis of induction: $n = 2$ and it is obvious.

Hypothesis of induction: The theorem is true $n \leq k. (k \geq 2)$

Assertion of Induction: The theorem is true $n = k + 1$.

Proof of induction: let T be a tree that has a vertical of $k + 1$ and let A be an arbitrary matrix of T. T has two vertices at least. We have chosen a pendant vertex vi which is not the vertex of reference A and the edge and which is the case on v_i. Then,

$$a_{it} = (\pm)1 \quad \text{and} \quad a_{ij} = 0, \text{ when } j \neq t.$$

We're expanding |A| by the i_{th} row:

$$|A| = (\pm)(-1)^{i+t}|A'|,$$

Where A′ is a minor equivalent to a_{it}. We're writing $T' = T - v_i$ That's also a tree (v_i is a vertex pendant). We use the induction assumption to obtain $|A'| = \pm 1$ because A′ is obviously an incidence matrix of T ′.

Corollary If the digraph G contains no loops; an all-vertex frequency matrix is μ(G).

Proof. If the all-vertex matrix rows or columns are rearranged, the matrix rank does not change. Reorganize the vertices and arcs according to components. The all-vertex incidence matrix is then a block diagonal matrix with an all-vertex incidence matrix of a component.

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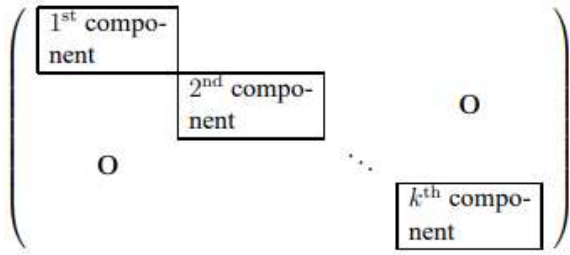
ni is the number of vertices in the i_{th} part. Each part is a spanning tree whose incidence matrix by Theorem 1 has a non-zero determinant. The all-vertex incidence matrix of the i_{th} is obtained by adding columns and a row to the matrix incidence of the respective tree. The added row would rely linearly on other rows such that the rank of this matrix is the same as that of the matrix (= $ni − 1$). Note that the rank is 0 = 1 − 1 when a variable is trivial in the special case. Thus, the
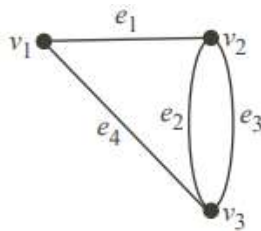
Rank of A = total of the part ranks

$$= (n_1 - 1) + \cdots + (n_k - 1)$$
$$= \underbrace{n_1 + \cdots + n_k}_{=n} - k = \rho(G)$$

Cut Matrix

If all the breaks of a non-trivial loop are I1, , The cut matrix of G is then t × m matrix Q = (qij), where m is the number of edges of G and

$$q_{ij} = \begin{cases} 1 \text{ if } e_j \in I_i \text{ (the cut is interpreted as an edge set)} \\ 0 \text{ otherwise.} \end{cases}$$

the cuts are $I_1 = \{e1, e4\}$, $I_2 = \{e2, e3, e4\}$ and $I_3 = \{e1, e2, e3\}$. The cut matrix is

$$\mathbf{Q} = \begin{array}{c} \\ \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \end{array} \begin{array}{c} \\ I_1 \\ I_2 \\ I_3 \end{array}$$

Remark. If the graph has n vertices, then it has $\frac{1}{2}(2^n - 2) = 2^{n-1} - 1$ Breaks. Cuts. Typically, not many distinct edge sets are present. We just take one cut for the cut matrix referring to a collection of edges such that there are no replicated rows. Even, generally there are so many rows.

If G is a loop less and nontrivial digraph, we add to each cut an arbitrary direction: orientation *(V1, V2)* is *V1 to V2*. In other terms, we consider oriented cuts and select just one path from the two choices. The cut matrix $Q = (q_{ij})$ is then

$$q_{ij} = \begin{cases} 1 \text{ if } e_j \in I_i \text{ and they are in the same direction} \\ -1 \text{ if } e_j \in I_i \text{ and they are in opposite directions} \\ 0 \text{ otherwise.} \end{cases}$$

For the digraph
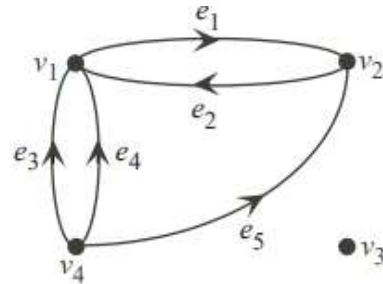
$I_1 = \{e1, e2, e3, e4\}$ in the direction of e1), $I_2 = \{e3, e4, e5\}$ (in the e3 direction), $I_3 = \{e1, e2, e5\}$ (in the e1 direction) and I4 = ∅. in the direction of e4. The matrix is sliced

$$\mathbf{Q} = \begin{array}{c} e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \\ \begin{pmatrix} 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} I_1 \\ I_2 \\ I_3 \\ I_4 \end{array}$$

Since $\langle\{v\}, V - \{v\}\rangle$ The cut for each vertex v, the all-vertex matrix incidence rows are Q rows. If you are working with directed graphs, you can have to multiply such rows by −1.

Theorem 2. Each column of the cut matrix of a digraph can be interpreted in two different ways as a linear combination of the all-vertex incidence matrix. Correlation coefficient non-zero are = +1 or all = −1.

Proof. Enable Q to be the cutting matrix of a digraph $G = (V, E)$ and let A be the all-vertex matrix. Let *(V1, V2)* be the cut corresponding to Q in the I th row (note that it is oriented). If required reindexing, we may presume that

$$V_1 = \{v_1, \ldots, v_r\} \quad \text{and} \quad V_2 = \{v_{r+1}, \ldots, v_n\}.$$

We write

$$\mathbf{q}_i = i^{\text{th}} \text{ row of } \mathbf{Q} \quad \text{and} \quad \mathbf{a}_t = t^{\text{th}} \text{ row of } \mathbf{A}.$$

We show that

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$$q_i = \sum_{t=1}^{r} a_t = - \sum_{t=r+1}^{n} a_t,$$

which proves the theorem. Let $(v_p, v_q) = e_k$ be the kth arc of G. Then,

$a_{pk} = k^{th}$ element of the vector $a_p = 1$,

$a_{qk} = k^{th}$ element of the vector $a_q = -1$

And

$$a_{jk} = 0 \quad \text{if} \quad j \neq p, q.$$

We get four cases:

- $v_p \in V_1$ and $v_q \in V_2$: Now $p \leq r$ and $q \geq r+1$ so $q_{ik} = 1$ and

$$q_{ik} = \sum_{t=1}^{r} a_{tk} = - \sum_{t=r+1}^{n} a_{tk}.$$

- $v_p \in V_2$ and $v_q \in V_1$: Now $p \geq r+1$ and $q \leq r$ so $q_{ik} = -1$ and

$$q_{ik} = \sum_{t=1}^{r} a_{tk} = - \sum_{t=r+1}^{n} a_{tk}.$$

- $v_p \in V_1$ and $v_q \in V_1$: Now $p \leq r$ and $q \leq r$ so $q_{ik} = 0$ and

$$q_{ik} = \sum_{t=1}^{r} a_{tk} = - \underbrace{a_{r+1,k}}_{=0} - \cdots - \underbrace{a_{nk}}_{=0}.$$

- $v_p \in V_2$ and $v_q \in V_2$: Now $p \geq r+1$ and $q \geq r+1$ so $q_{ik} = 0$ and

$$q_{ik} = \underbrace{a_{1k}}_{=0} + \cdots + \underbrace{a_{rk}}_{=0} = - \sum_{t=r+1}^{n} a_{tk}.$$

The above statements are true for all k.

Example. (Continuing from the previous example) The corresponding row of l1 is

(1, −1, −1, −1, 0) = (1, −1, −1, −1, 0) = −(−1, 1, 0, 0, −1) − (0, 0, 0, 0, 0) − (0, 0, 1, 1, 1).

Corollary. The rank of the diagram G matrix is Ś(G).

Proof. Vector A of G is indeed a submatrix of the cut matrix Q of G. And (by Theorem 1 Corollary)

$$\text{rank}(\mathbf{Q}) \geq \text{rank}(\mathbf{A}) = \rho(G).$$

On the other side, each Q row can be represented as a linear combination of the A rows in Theorem 2. Therefore,

$$\text{rank}(\mathbf{Q}) = \text{rank}(\mathbf{A}) = \rho(G).$$

The effect is that the cut matrix Q can be represented as

$$\mathbf{Q} = \mathbf{A_1 A},$$

Where A1 elements are either 0 or ±1. The A1 matrix can also be built from the method in the Theorem 2 argument.

If the graph G is connected, it includes a T-covering tree and a related simple break. The simple cuts are often cutting (when cuts are viewed as boundary sets). The cut matrix Q of G therefore has a submatrix Q_f that corresponds to these simple cut sets. This matrix is considered the essential matrix of the cuts. In a similar manner, the related digraph G has a simple matrix collection: if a basic cut is viewed as a set the path of the cut would be the same as that of the corresponding branch of T. If the edges of G are substituted such that the divisions are first and the basic cut sets are ordered in the same order, the basic matrix comes in the form

$$\mathbf{Q_f} = \left(\begin{array}{c|c} \mathbf{I}_{n-1} & \mathbf{Q_{fc}} \end{array} \right),$$

Where $\mathbf{I}_{n-1}$ Is the n − 1 row identity matrix? The rank of Q_f is therefore $n - 1 = \rho(G)$.

Example. (We took out the previous example's vertex v3) so we get a related digraph. We chose the spanning tree
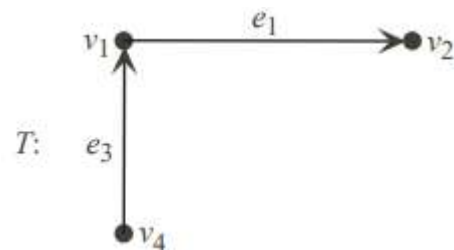
The basic cut sets are l_2 = {e3, e4, e5} (in e3) and l_3 = {e1, e2, e5} (in e1). And, Then Then

$$\mathbf{Q_f} = \begin{array}{ccccc} e_1 & e_3 & e_2 & e_4 & e_5 \end{array} \\ \left(\begin{array}{cc|ccc} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{array} \right) \begin{array}{c} I_3 \\ I_2 \end{array}$$

$$\mathbf{Q} = \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \left(\begin{array}{ccccc} 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 1 \end{array} \right) \begin{array}{c} \\ \leftarrow \\ \leftarrow \end{array}$$

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Circuit Matrix

The loop less graph $G = (V, E)$ comprising circuits is used. We mention circuits. , $G: C1, . , C\ell$. The matrix of the circuit G is $\ell \times m$ matrix $B=(b_{ij})$

$$b_{ij} = \begin{cases} 1 \text{ if the arc } e_j \text{ is in the circuit } C_i \\ 0 \text{ otherwise} \end{cases}$$

(as usual, $E = \{e1, . . ., em\}$).

The circuits in digit G are oriented, i.e. each circuit is driven randomly in order to determine the circuit matrix. After the orientations have been selected, the G circuit matrix is $B = (b_{ij})$.

$$b_{ij} = \begin{cases} 1 \text{ if the arc } e_j \text{ is in the circuit } C_i \text{ and they in the same direction} \\ -1 \text{ if the arc } e_j \text{ is in the circuit } C_i \text{ and they are in the opposite direction} \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 3. For a digraph, $\mathbf{BQ^T = O}$ (zero matrix).

Proof. In the previous theorem, half of the non-zero numbers for each part of the BQ^T in the dot product are +1. The other non-zero numbers are −1. The dot product is thus = 0.

If at least one circuit is found in digraph G, the rank of its matrix B is $\mu(G)$. Further, if G is associated then the B circuit matrix can be represented as $B = B_2B_f$, in which 0's and ±1's compose of matrix B_2, and the Q cut out matrix may be expressed as $Q = Q_1Q_f$, in which 0's and ±1's are the Q_1 matrix.

Proof. First, when G is associated, we consider the event. We pick a spanning tree T by G and arrange the m edges of G such that the T branches arrive first and the T *. We sort the simple cut sets in the same order as the divisions and ties. And, Then,

$$\mathbf{Q_f = (I_{n-1} \,|\, Q_{fc})} \quad \text{and} \quad \mathbf{B_f = (B_{ft} \,|\, I_{m-n+1})}.$$

The B blocks may be designed in the same way:

$$\mathbf{B = (B_1 \,|\, B_2)}$$

Because Q_f is a semi-matrix of Q and Bf is a sub-matrix of B, Theorem 3 notes that

$$\mathbf{O = B_f Q_f^T = (B_{ft} \,|\, I_{m-n+1})(I_{n-1} \,|\, Q_{fc})^T = (B_{ft} \,|\, I_{m-n+1})\left(\frac{I_{n-1}}{Q_{fc}^T}\right)}$$

$$\mathbf{= B_{ft}I_{n-1} + I_{m-n+1}Q_{fc}^T = B_{ft} + Q_{fc}^T.}$$

Hence

$$\mathbf{B_{ft} = -Q_{fc}^T.}$$

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Furthermore, since Q_f is a submatrix of Q, we can use the same theorem to get

$$\mathbf{O = BQ_f^T = (B_1 \,|\, B_2)(I_{n-1} \,|\, Q_{fc})^T = (B_1 \,|\, B_2)\left(\frac{I_{n-1}}{Q_{fc}^T}\right)}$$

$$\mathbf{= B_1I_{n-1} + B_2Q_{fc}^T = B_1 - B_2B_{ft}.}$$

Hence

$$\mathbf{B = (B_2B_{ft} \,|\, B_2) = B_2(B_{ft} \,|\, I_{m-n+1}) = B_2B_f,}$$

As claimed. Likewise, Q can be represented as $Q = Q_1Q_f$ as claimed, which is simple anyhow, provided that the Q rank is n −1, with 0's and ±1's components. Each B set is a linear combination of the ranks that fit the simple circuits, and the rank B is at most equivalent to the rank Bf $\geq m - n + 1$. On the other side, as already stated, the grade B is − m − n + 1. Thus, for a related digraph, rank(B) = m − n + 1 (= $\mu(G)$).

When a digit G (with at least 1 circuit) is disconnection, it is divided into components (k ≥ 2 components) and circuit B is divided into blocks of components (compare the corollary proof of Theorem 1).

$$\text{rank}(\mathbf{B}) = \sum_{i=1}^{k}(m_i - n_i + 1) = m - n + k = \mu(G).$$

Note that the data often includes the formula, $\mathbf{B_{ft} = -Q_{fc}^T}$, which connects the basic cut matrix with the basic circuit matrix.

Matrices over GF (2) and Vector Spaces of Graphs

The set {0, 1} is classified as a field if the addition and multiplications are described as follows (i.e. the same arithmetical rules as real numbers):

+	0	1
0	0	1
1	1	0

×	0	1
0	0	0
1	0	1

−1 = 1 and 1 −1 = 1 in any situation. This is the GF (2) sector.

Theorems 1 ,2, 3, and their corollaries often refer to "undirected graphs" if we consider elements 0 and1 of the total vertex occurrence, decomposition, fundamental cut, circuit and simple circuit matrices for a ("undirected") graph. (Note that the proofs are the same in area GF (2). The evidence is the same.

Vector space for "undirected" graphs is above field GF (2). For graph theory (i.e. over field R) the vectors spaces are real. The area of the line of the cut matrix

is the space of the cut. Similarly, the circuit space is the row space of the circuit matrix. The component of the split area is the (di)graph and the circle space component is the (di)graph's nullity. In comparison, orthogonal complements are the space cut and the circuit space. (All these assertions are explicitly based on the following results.)

We also use subgraphs to manage the above spaces, i.e. we define a vector with the subgraph formed by the respective arcs. With "undirected" diagrams, inserting GF (2) vectors refers to the activity of the ring sum.

CONCLUSION

It turns out that much graph knowledge is contained in these matrices. We may measure stuff from such graphs such as the number of spanning trees, the algebraic connectivity, etc. Most of these are well described problems in terms of their own value and thus can be computational. In this article we address the matrices and vector spaces in charts.

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