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Functions of Bounded Variation

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Abstratct – In this paper we will discuss the concept of the functions of bounded variation. It has much use in Mathematics. It is much linked with the concept of 'Monotonic Functions.' Let us consider a closed and bounded interval [a, b]. Let P be the partition of [a, b] where P is { x_0, x_1, …… , x_n} such that

$$a = x0 < x1 < x2 < \cdots \ldots. < xn = b.$$

This partition P of [a, b] divides [a, b] into n sub intervals [x_0, x_1], [x_1, x_2], ….. , [x_{n-1}, x_n]

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DEFINITION :

Let [a, b] be a closed and bounded interval and $f:[a,b] \to R$ be a function. Let P be the partition of [a, b] where P is given by $a=x_0<x_1<x_2<\ldots\ldots<x_n=b$

Let us consider the sum

$$V(f,P) = |f(x_1)-f(x_0)| + |f(x_2)-f(x_1)| + ---+ |f(x_n)-f(x_{n-1})|$$

As we take different partitions P, we have a set of non negative real numbers given by V(f, P). In case the set of these non negative real numbers given by V(f, P) is bounded above, then we say that f is a function of bounded variation on [a, b].

We can also call it a BV-function on [a, b].

The least upper bound of the set mentioned above is called the total variation of f on [a, b]. We denote it by V(f)[a, b]

From definition it follows that V(f)[a, b]=0 if and only if f is a constant function on [a, b].

For example:

1. Let us consider f(x)=x on [a, b] For any partition P:a = $x_0<x_1<$ - - - - - $<x_n=b$ of [a, b] V(f, P) = b-a.

For different partitions P: sup V(f, P)= b-a. P

Thus, f is a function of bounded variation on [a, b] and V(f) [a, b] = b-a.

2. Let f(x) = Sinx on [a, b]

For P= {a=$x_0<x_1<$ - - - - $<x_n$=b} partition of [a, b]

$$V(f,P) = |\sin x_1 - \sin x_0| + |\sin x_2 - \sin x_1| + ---+ |\sin x_n - \sin x_{n-1}|$$
(2)

The expression on the right hand side of above equation is less than or equal to b-a by mean value theorem.

This holds for every partition P of [a, b]. Therefore it follows that f is a function of bounded variation on [a, b].

The concept of **'functions of bounded variation'** is closely related to the concept of **'Monotonic functions'** which follows from the following Theorem.

Theorem 1: Let f defined on [a, b] be a real valued monotonic function on [a, b]. Then f is a function of bounded variation on [a, b].

Proof: Case 1. Let f is a monotonically increasing function on [a, b]

Consider a partition P of [a, b] as follows:

P : a = $x_0<x_1<$ - - - - - $<x_n$=b be the partition of [a, b]

By definition

V(f, P) =

V(f, P) = $|f(x_1)-f(x_0)| + |f(x_2)-f(x_1)| + ----+|f(x_n)-f(x_{n-1})| = f(b)-f(a).$

which is true for any partition P.

Sup. V(f, P) = f(b)-f(a)

P

Thus f is a function of bounded variation on [a, b].

Case 2. is similar when f is a monotonically decreasing function for which V(f, P) = f(a) – f(b)

NOTE 1. A function of bounded variation on [a, b] is bounded.

NOTE 2. A bounded function need not be of bounded variation. For example: f on [0, 1] defined by

$$f(x) = \begin{cases} x\cos\frac{\pi}{2x} & x\ not\ equal\ to\ zero \\ 0 & x = 0 \end{cases}$$

Definition : Lipschitz Function :

Let f is a real valued function defined on [a, b]. Then f is said to be a lipschitz function on [a, b] if there exist a positive real number M such that it satisfies the following condition

$$|f(x_1) - f(x_2)| \le M|x_1 - x_2|$$

where x_1, x_2 **Є**[a, b]

(3)

Since for a Lipschitz function f on [a, b]

$$|f(x_r) - f(x_{r-1})| \le M|x_r - x_{r-1}|$$

r=1, 2, ----, n

where $a= x_0<x_1< - - - < x_n = b$ is the partition P of [a, b], consequently, V(f, P) ≤ M(b-a)

Hence, a **Lipschitz function** is always a function of bounded variation. But not conversely. Like f(x) = $\sqrt{x}$ on [0, 1] being a monotonically increasing function on [0, 1] is a function of bounded variation but is not Lipschitz.

NOTE 3. A continuos real valued function on [a, b] with bounded derivative on (a, b) is a function of bounded variation on [a, b]. Since there exist a positive real number M such that $|f'(x)| \le M$ for all x**Є**(a, b) and consequently, V(f, P) ≤ M(b-a) from which it follows that f is a function of bounded variation on [a, b].

But boundedness of derivative off is not a necessary condition for a function to be of bounded variation. For example f(x) = $\sqrt{x}$, x **Є** [0, 1] being monotonically increasing function on [0, 1] is of bounded variation whereas it's derivative is not bounded on (0, 1).

NOTE 4. Let us consider a continuous function f defined on a closed and bounded interval [0, 1] as :

$$f(x) = \begin{cases} x\cos\frac{\pi}{2x} & ;\quad x \in (0,1] \\ 0 & x = 0 \end{cases}$$

But f is not a function of bounded variation. Thus a continuous function need not be a function of bounded variation.

NOTE 5. (i) Sum of two functions of bounded variation say f and g is again a function of bounded variation. Since V(f+g) [a, b] ≤ V(f) [a, b] + V(g) [a, b].

(ii) Scalar multiplication of a function of bounded variation is again a function of bounded variation since V(cf)[a, b] ≤ |c| V(f)[a, b] for any c **Є** R

(iii) From (i) and (ii) we can conclude that the set of functions of bounded variation forms a vector space with respect to the addition and scalar multiplication defined above in (i) and (ii) over R.

NOTE 6. Let f and g are two real valued functions defined on [a, b] such that f and g both are functions of bounded variation on [a, b]. Then V(fg)[a, b] ≤ AV(g) [a, b] + BV(f) [a, b]

where A and B are defined as follows:

A = l.u.b {|f(x)| : x **Є** [a, b]}

(4)

B = l.u.b {|g(x)| : x **Є** [a, b]}

Hence f.g is also a function of bounded variation.

NOTE 7. If f is a real valued function defined on [a, b] such that it is a function of bounded variation on [a, b] and there is a positive real number M such that f(x)≥M for all x Є [a, b], Then 1/f is also a function of bounded variation on [a, b] and

$$V\left(\frac{1}{f}\right)[a,b] \le \frac{1}{M^2}V(f)[a,b]$$

NOTE 8. For f a real valued function defined on [a, b], if f is a function of bounded variation on [a, b] then so is |f|.

Definition: **Refinement of a Partition**.

Let P be the partition of a closed and bounded interval [a, b] where P:a = $x_0<x_1<$ --- $<x_n$ = b. A partition P_1 of [a, b] is a refinement of P if P_1 is obtained by adjoining a finite number of additional points to P.

NOTE 9. A function f of bounded variation on [a, b] and P being the partition of [a, b] and P_1 being its refinement has the following property.

$V(f, P_1) \geq V(f, P)$

NOTE 10. Let f be a real valued function defined on closed and bounded interval [a, b] and c Є (a, b) and f being a function of bounded variation on [a, b] implies V(f) [a, b] = V(f) [a, c] + V(f) [c, b] and we have the following :

(i) f is a function of bounded variation on [a, c].

(ii) f is a function of bounded variation on [c, b].

NOTE 11. Let f be defined on [a, b]. f is a function of bounded variation on [a, b] if it is possible to divide [a, b] into subintervals which are finite in number, such that f is monotone on each of these subintervals. And moreover,

V(f) [a, b] = V(f) $[x_0, x_1]$ + V(f) $[x_1, x_2]$ + - - - - - + V(f) $[x_{n-1}, x_n]$

where a = $x_0<x_1<$ - - - - $<x_n$ = b

NOTE 12. We take f:[a, b] → R a function of bounded variation on [a, b] and another function φ : [a, b] → R which is :

(i) bounded on [a, b], (5)

(ii) f(x) = φ(x) for all x in [a, b] except at finitely many points in [a, b]. Then φ is also a function of bounded variation on [a, b].

Definition : Variation Function : For a function f defined and of bounded variation on [a, b] we define a new function V on [a, b] as : V(a) = 0 and V(x) = V(f)[a, x] where $a<x\leq b$ This V is called variation function of f on [a, b]. This function V is a monotonically increasing function on [a, b] and moreover we have the following results:

(i) V+f is a monotonically increasing function on [a, b].

(ii) V-f is a monotonically increasing function on [a, b].

Moreover, f = (V+f)-f

Thus, f can be expressed as the difference of two monotonically increasing functions namely V+f and f. But this representation is not unique since f = V-(V-f) where V and V-f are both monotonically increasing functions.

NOTE 13. The points of discontinuities of a function f : [a, b] → R, where f is a function of bounded variation, is atmost countable since f being a function of bounded variation is expressible as the difference of two monotonically increasing functions and set of discontinuities of a monotonic function forms a countable set.

NOTE 14. Let f be a function of bounded variation on [a, b] and V be the corresponding variation function on [a, b]. Then f is continuous at a point c in [a, b] if and only if V is continuous at c.

Consequently; f = (V+f)-V implies that f if expressible as the difference of two monotonic continuous functions. But this representation is not unique since f = V-(V-f)

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