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An Analysis upon the Asymptotic Structure of Banach Spaces: A Case Study of Envelope Functions

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Abstract – This research studies the asymptotic structures of Banach spaces through the notion of envelope functions. Analogous to the original ones, a new notion of disjoint-envelope functions is introduced and the properties of these functions in connection to the asymptotic structures are studied. The main result gives a new characterization of asymptotic- ℓ_p ***spaces in terms of the*** ℓ_p ***-behavior of disjoint-permissible" vectors of constant coefficients.***

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INTRODUCTION

Asymptotic structures of infinite-dimensional Banach spaces, introduced in the study of B. Maurey et al. (1994), reflect the behavior at infinity of finite-dimensional subspaces which repeatedly appear everywhere and far away in the space and are arbitrarily spread out along, for instance, a basis. This approach to infinite-dimensional spaces serves as a bridge between finite-dimensional and infinite-dimensional theories, in view of the outstanding developments in the Banach space theory in the 1990?s. For example, asymptotic-ℓ_p spaces were discovered in V.D. Milman and N. Tomczak-Jaegermann (1993) in connection with the distortion problem; and the game approach used in the study of B. Maurey et al. (1994) to define asymptotic structures originated by W.T. Gowers(2002). For these and many other aspects of asymptotic approaches to infinite-dimensional Banach spaces theory we refer the reader to the exhaustive survey by E. Odell (2002).

In its simplest form, the asymptotic structure of a Banach space is defined as follows. Given a Banach space *X* with a monotone basis, an *n*- dimensional space *E* with a monotone basis $\{e_i\}_{i=1}^n$ is an *asymptotic space* for *X* if there exists a finitely supported normalized vector y_1 *(block)* with support arbitrarily far along the basis $\{x_i\}$, then a normalized block y_2 with support arbitrarily far after the support, of y_1, then a normalized block y_3 with support arbitrarily far after the support of y_2 and so on, such that the blocks $y_1, \ldots, y_n$ obtained after *n* steps have behavior as close to the behavior of $\{e_i\}_{i=1}^n$ as we wish. (This means that any linear combination of $\{y_i\}_{i=1}^n$ has norm in *X* arbitrarily close to the norm in *E* of the corresponding linear combination of $\{e_i\}_{i=1}^n$.) The normalized blocks $y_1, \ldots, y_n$ are called *permissible* vectors. The set of all n-dimensional asymptotic spaces of *X* will be denoted by $\{X\}_n$. The asymptotic structure of *X* consists of all $E \in \{X\}_n$, for all $n \in \mathbb{N}$.

A Banach space *X* is called an *asymptotic-* ℓ_p *space* if there exists a constant $C \geq 1$ such that for all *n* and $E \in \{X\}_n$, the basis in *E* is C-equivalent to the unit vector basis of ℓ_p^n (for the precise definition see below). That is, asymptotic-ℓ_p spaces have only one type of asymptotic spaces. In [MMT], it is shown that for $1 < p < \infty$, if for all *n* and $E \in \{X\}_n$, *E* is C-isomorphic to ℓ_p^n, then *X* is asymptotic-ℓ_p. This means that in such situations, there is a natural isomorphism between *E* and ℓ_p^n, which is the equivalence between respective bases. Up to a constant, for $1 < p < \infty$, asymptotic-ℓ_p spaces have a *unique* asymptotic structure, and this in fact characterizes asymptotic-ℓ_p spaces.

The notion of disjoint-envelope functions is a convenient tool for studying asymptotic structures of spaces with an unconditional basis (or more generally, with asymptotic unconditional structure).

In general, asymptotic methods in the theory of infinite-dimensional Banach spaces rely on stabilizing information of finite nature \at infinity". This way we discard properties which may appear sporadically in the space and could be removed by passing to appropriately chosen subspaces or some other substructures. First methods of this kind began to develop in the 1970's, in connection with Ramsey theorems and the notion of a spreading model (to be described later in this introduction). The ideas behind what is now called

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an asymptotic theory of (infinite-dimensional) Banach spaces were crystallized in the early nineties in connection with spectacular developments of the infinite-dimensional Banach spaces.

From the very beginning of Functional Analysis initiated by the work of Banach in the 1920's the objective of the classical theory of infinite-dimensional spaces have been mainly to establish a structure theory for Banach spaces. Besides isomorphism type questions, primarily, the problems were centered around seeking subspaces with `nice' structural properties in all Banach spaces.

GEOMETRY OF BANACH SPACES

In this study, all spaces are real *separable Banach* spaces and all subspaces are closed subspaces. By *X, Y,...* we usually denote infinite-dimensional Banach spaces; we reserve *E.* F,... to denote finite-dimensional Banach spaces.

The norm in *X* is denoted by ||.||x, or simply by ||.|| if there is no ambiguity. By *Bx* we denote the closed unit ball $\{x \in X: \|x\| \leq 1\}$, and by S_X the unit sphere $\{x \in X: \|x\| = 1\}$ of *X.*

Linear continuous maps between two Banach spaces *X* and *Y* are called *operators* and denoted by $T: X \to Y$. If *T* is an isomorphism between *X* and *Y,* the *isomorphism constant* C is defined by $C = \|T\|\|T^{-1}\|$ and in this case we write $X \overset{C}{\simeq} Y$, or simply $X \simeq Y$ if we do not want to specify the isomorphism constant. We will say that *X* and *Y* are C-isomorpliic or simply isomorphic.

For a set *E* in A, $\overline{\text{span}}[E]$ denotes the closed *linear span* of *E* in *X* and conv[£r the closed *convex hull* of *E.*

As examples of Banach spaces we shall often use the classical sequence spaces c_0, and ℓ_p $(1 \leq p \leq \infty)$. c_0 is the space of all real sequences $x = (a_n)$ with $\lim_{n\to\infty} a_n = 0$ with the norm $\|x\|_\infty = \sup_n |a_n|$. For any $1 \leq p < \infty$, ℓ_p is the space of real sequences $x = (a_n)$ with $\sum_n |a_n|^p < \infty$, and the norm $\|x\|_p = (\sum_{n=1}^\infty |a_n|^p)^{1/p}$. ℓ_∞ is the space of all bounded real sequences $x = (a_n)$ with the norm $\|x\|_\infty = \sup_n |a_n|$. For each $n \in \mathbb{N}$, ℓ_p^n $(1 \leq p \leq \infty)$ denotes the n-dimensional space $\mathbb{R}^n$ with ℓ_p*-norm.*

DISJOINT ENVELOPE FUNCTIONS

Let *X* be a Banach space with an asymptotic unconditional structure (with a constant $C \geq 1$). We define the set $\{X\}^d$ of all normalized *disjoint-permissible vectors* in *X* as follows. For $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \in \{X\}^d$ if there exist $\{e_j\}_{j=1}^m \in \{X\}_m$ for some $m \geq n$ and a disjoint partition $\{A_1, A_2, \ldots, A_n\}$ of $\{1, 2, \ldots, m\}$ such that for each $1 \leq i \leq n$, $x_i = \sum_{j \in A_i} \alpha_j e_j$ for some scalars $\alpha = (\alpha_j)$ such that $\|x_i\| = 1$.

First we make a few remarks about the set $\{X\}^d$ (where superscript *d* stands for 'disjoint'). Clearly, for all *n e* N and $\{e_i\}_{i=1}^n \in \{X\}_n$ we have that $\{e_i\}_{i=1}^n \in \{X\}^d$. i.e., $\bigcup_n \{X\}_n \subset \{X\}^d$. If $\{x_i\} \in \{X\}^d$, then $\{x_i\}$ is an unconditional basic sequence (with constant *C).* It is also clear that if $\{u_j\}$ is a block (successive or just disjoint) basis of some $\{x_i\} \in \{X\}^d$, then $\{u_j\} \in \{X\}^d$ as well. Finally, if $\{x_i\}_{i=1}^n \in \{X\}^d$ then $\{x_{\pi(i)}\}_{i=1}^n \in \{X\}^d$, where π is a permutation of $\{1, \ldots, n\}$. This property, obviously, is not shared, in general, by the bases of asymptotic spaces.

We also have the following property of $\{X\}^d$ which is inherited from $\{X\}_n$. If $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are in $\{X\}^d$, then there exists $\{z_i\}_{i=1}^{n_1+n_2} \in \{X\}^d$ such that $\{z_i\}_{i=1}^{n_1} \overset{1}{\sim} \{x_i\}_{i=1}^{n_1}$ and $\{z_i\}_{i=n_1+1}^{n_1+n_2} \overset{1}{\sim} \{y_i\}_{i=1}^{n_2}$.

Indeed, if $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are disjoint blocks of the bases $\{e_i\}_{i=1}^{m_1}$ and $\{f_i\}_{i=1}^{m_2}$ of some asymptotic spaces respectively, then we can find an asymptotic space $\{g_i\}_{i=1}^{m_1+m_2}$ such that $\{e_i\}_{i=1}^{m_1} \overset{1}{\sim} \{g_i\}_{i=1}^{m_1}$ and. Hence the corresponding disjoint blocks $\{z_i\}_{i=1}^{n_1+n_2}$ of $\{g_i\}_{i=1}^{m_1+m_2}$ have the desired property. When $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are in $\{X\}^d$, to avoid repetitions, we will simply say that $\{x_i, y_i\} \in \{X\}^d$ without referring to $\{z_i\}$.

We define now the natural analogs of envelope functions on $\{X\}^d$.

Definition 1 Let X be a Banach space with, an asymptotic unconditional structure. For $a = (a_i) \in c_{00}$, let $g_X^d(a) = \inf \|\sum_i a_i x_i\|$ and $r_X^d(a) = \sup \|\sum_i a_i x_i\|$,

where the inf and the sup is taken over all $\{x_i\} \in \{X\}^d$. We call g_X^d and r_X^d the

disjoint-lower and disjoint-upper-envelope functions respectively.

It is easy to see that both functions g_X^d and r_X^d are 1-symmetric and 1- unconditional. Moreover, while r_X^d defines a norm on e_0o, g_X^d satisfies triangle inequality on disjointly supported vectors (of c_{00}).

Indeed, let $a = (a_i)$ and $b = (b_i)$ be two disjoint vectors in coo and let $\varepsilon > 0$ be arbitrary. Pick $\{x_i\}$ and $\{y_i\}$ in $\{X\}^d$ such that $g_X^d(a) + \varepsilon/2 \geq \|\sum_i a_i x_i\|$ and $g_X^d(b) + \varepsilon/2 \geq \|\sum_i b_i y_i\|$. Then, by the above remark, $\{x_i, y_i\} \in \{X\}^d$ and hence

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$$\begin{aligned} g_X^d(a+b) &\leq \|a_1x_1+b_1y_1+a_2x_2+b_2y_2+\dots\| \\ &\leq \|a_1x_1+a_2x_2+\dots\|+\|b_1y_1+b_2y_2+\dots\| \\ &\leq g_X^d(a)+g_X^d(a)+\varepsilon. \end{aligned}$$

Since $\varepsilon>0$ was arbitrary, it follows that $g_X^d(a+b)\leq g_X^d(a)+g_X^d(b),$ whenever $a,b\in c_{00}$ are disjointly supported.

To compare the disjoint-envelope functions with the original envelopes, note that for all $a\in c_{00},$ by the definition of these functions we have, $g_X^d(a)\leq g_X(a)\leq r_X(a)\leq r_X^d(a).$

We will use the following convenient notation. Let (e_i) be the unit vector basis of $c_{00}.$ For $a=(a_i)\in c_{00},$ occasionally we will write $g_X^d(\sum_i a_ie_i)$ instead of $g_X^d(a).$ Moreover, for any finite number of successive vectors $b^i=(b^i_j)\in c_{00}$ such that $g_X^d(b^i)=1$ for $i=1,2,\dots$ and for any vector $a=(a_i)\in c_{00},$ we write $g_X^d(\sum_i a_ix_i)$ instead of $g_X^d(\sum_i a_ib^i),$ where $x_i=\sum_j b^i{}_je_j$ are blocks of the basis (e_i) of c_{00} normalized with respect to $g_X^d.$ We'll use similar notation for r_X^d as well.

ASYMPTOTIC STRUCTURE OF BANACH SPACES

Let A be an infinite dimensional Banach space. We want to look at finite dimensional subspaces which occur 'at infinity'. The picture that the reader should have in mind is that of a Banach space with a basis, and the asymptotic spaces as being those finite block subspaces which occur arbitrarily far down the basis.

We will define asymptotic spaces using a vector game between two players V and S. Let $\mathcal{B}(X)$ be a family of infinite dimensional subspaces of *X* satisfying the filtration condition

$$F,G\in\mathcal{B}(X)\Rightarrow\exists H\in\mathcal{B}(X)\text{ s.t. }H\subseteq F\cap G.$$

For example, we could take $\mathcal{B}(X)$ to be the finite co-dimensional subspaces of A, or if A has a basis we could take the family of tail subspaces. Throughout this study we will be looking at Banach spaces with a basis, so we will assume that the family $\mathcal{B}(X)$ is the collection of tail subspaces from now onwards.

The game is played as follows. S chooses a subspace $X_1\in\mathcal{B}(X).$ V responds by choosing a normalized vector $x_1\in X_1.$ S then chooses a subspace $X_2\subseteq X_1$ with $X_2\in\mathcal{B}(X).$ V then responds by choosing a normalized vector $x_2\in X_2$ such that $\{x_1,x_2\}$ is basic with basis constant $\leq 2.$ The game continues in this vein until the nth turn, where S selects a subspace $X_n\in\mathcal{B}(X)$ with $X_n\subseteq X_{n-1},$ and V chooses a normalized vector $x_n\in X_n$ such that $\{x_1,\dots,x_n\}$ is basic with basic constant $\leq 2.$

Given a finite dimensional Banach space *E* with basis $(e_i)_{i=1}^n$ with basis constant $\leq 2,$ and $\varepsilon>0,$ we say that V wins the game for *E* and $\varepsilon,$ if the vectors $\{x_1,\dots,x_n\}$ are $(1+\varepsilon)$- equivalent to $(e_i)_{i=1}^n.$ We will call *E* an asymptotic space of A, if V has a winning strategy for the game for *E* and every $\varepsilon>0.$ The collection of all 71-dimensional asymptotic spaces for A will be denoted by $\{X\}_n.$

It is a consequence of Krivine's theorem that there exists $p\in[1,\infty]$ such that for every n, ℓ_p^n (with its standard basis) is in $\{X\}_n.$ Hence, the simplest possible asymptotic structure that can occur is where the basis $(e_i)_{i=1}^n$ of every $E\in\{X\}_n$ is C-equivalent to the standard basis of ℓ_p^n for some *C* independent of n. We will call such spaces asymptotic ℓ_p spaces.

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