Rational Numbers

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Abstract – The aim here is to discuss some important concepts about rational numbers like the algebraic properties of rational numbers, the order properties of rational numbers, the density property of rational numbers along with the geometrical representation of rational numbers.

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DEFINITION:

A Number of the form p/q where p and q are integers and q≠0 is called a rational number. The set of rational numbers is denoted by Q.

We firstly discuss the algebraic properties of Q. Addition is defined on Q and it satisfies the following properties.

1. The **closure** property :

For all a, b Є Q, a + b Є Q

2. The **associative** property :

For all a, b, c Є Q, a + (b + c) = (a + b) + c

3. Existence of **Identity** element :

For all a Є Q, there exist 0 in Q s.t.

a + 0 = a = 0 + a.

4. Existence of **Inverse** Element :

For an element a in Q we have a corresponding element –a in Q such that

a + (-a) = 0 = (-a) + a

5. **Commutativity** :

For all a, b Є Q, a + b = b + a.

Multiplication is also defined on Q and it satisfies the following properties :

1. The Closure property :

For a, b Є Q, a . b Є Q

2. Associative property :

For a, b, c Є Q, (a . b) . c = a . (b . c) (2)

3. Existance of Identity element :

For a Є Q we have 1 Є Q such that a . 1 = a = 1 . a

4. Existance of Inverse element :

For 0 ≠ a Є Q we have $\frac{1}{a} \in Q$ such that

$$a.\frac{1}{a} = \frac{1}{a}.a = 1$$

5. Commutativity :

For a, b Є Q we have a . b = b . a

We now discuss the order properties of Q : We define a relation < on Q as follows:

For a, b Є Q, a<b if a is less than b. This relation on Q satisfies the following conditions:

Property 1. Law of Trichonomy:

For a, b Є Q exactly one of the following condition holds

a<b or a>b or a=b.

Property 2. Trasitivity :

a<b and b<c then a<c.

Property 3. a<b and a, b>0

a+c < b+c for a, b, c Є Q

Property 4. a<b and c>0

=> a.c < b.c for a, b, c Є Q.

The first property namely **law of trichonomy** also states that for any rational number a, a<0, a=0, or a>0

Definition: Ordered Field:

A field together with an order relation defined on it and satisfying the above four

properties is said to be an ordered field.(3)

Consequently, Q is an ordered field.

Next is about the density property of Q .

Let us consider two rational numbers x and y. where x < y. There exists a rational number r such that x < r < y. i.e. between any two rational numbers there exists a rational number.

Since x < y

therefore x + y < y + y (By property 3)

=> $\frac{1}{2}(x+y) < \frac{1}{2}(y+y)$ (By property 4)

=> $\frac{1}{2}(x+y) < y$

Again x < y

=> x + x < x + y (By property 4)

=> $x < \frac{1}{2}(x+y)$

Thus, we have

$$x < \frac{1}{2}(x+y) < y$$

Take

$$r = \frac{1}{2}(x+y)$$

Thus, we have a rational number r between two rational numbers x and y. Similarly we can find another rational number between x and $\frac{1}{2}(x+y)$. And by continuing this process indefinitely we can find infinitely many rational numbers between any two given rational numbers. This property is called denseness property of Q.

Geometrical Representation of Rational Numbers:

We can represent rational numbers by points on a straight line. Consider the line X'X. Take on it a point o such that o divides X'X into two parts. The right side part will be considered as positive side and the left side part is negative side. Consider a point A_1 on the right of o. Let o represents the rational number zero and A_1 represents the rational number one. Taking the distance oA as unit distance, we can represent each rational number by a unique point on the line X'X. Represent the positive integers 2, 3, 4, … by the points A_2, A_3, ….. on the right of o.

And so we have $oA_2 = 2.oA$

$oA_3 = 3.oA$ ……

The negative integers -1, -2, …… are represented by the points A_1', A_2', ….. which are on the left of o. And so we have $oA_1' = oA_1$

(4)

$oA_2' = oA_2 = 2 . oA$ …..

Now, we take a positive rational number say r, which is of the form p/q where p and q are positive integers. To represent this on X'X, we measure p times the distance oA_1, to the right of o to get a point B and then we measure q^{th} part of the distance 0B on the right side of o to get a point P. This point P will represent the rational number r. Let the negative of this rational number r be t i.e. t = -r, then a point P' on the left of o such that oP'=oP will represents t. In this way we can represent every rational number on the line X'X. By this method, if we plot all the rational numbers on this line then the whole line is covered by rational numbers.

Note 1. If we consider a point D on the right of o in such a way that oD is equal to the length of the diagonal of the square which is made on the side oA, then this D is not a rational number.

Note 2. There is no rational number r such that $r^2 = 2$.

Let r is a rational number. Take r=p/q where p and q are positive integers and gcd of p and q is 1.

Since $r^2 = 2$

Therefore

$$\left(\frac{p}{q}\right)^2 = 2$$

=> $p^2 = 2q^2$

=> p^2 is even => p is even.

Let p = 2m where m is any integer.

Therefore $p^2 = 2q^2$ implies $4m^2 = 2q^2$

=> $2m^2 = q^2$

and thus q^2 is even and so is q.

Now since p and q are both even so gcd of p and q cannot be 1 which is a contradiction to our assumption.

Therefore there is no rational number satisfying $r^2 = 2$.

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