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Cantor Set: Construction and Properties

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Abstract – Here we will discuss the construction and properties of Cantor Set. We need to use the ternary expansions of real numbers.

---------------------------------x---------------------------------

Let us take [0, 1]. We take this interval for the arithmetical characterization of Cantor Set. In the ternary expansion of a real number, say x, we will use the digits 0, 1 and 2. Hence, we can take $x = 0.a_1\, a_2\, a_3$ ----------- in the ternary scale.

i.e. $$x = \frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + ----$$

where a_1, a_2, a_3, ------- will take values 0, 1 and 2 only.

Numbers like

$$\frac{1}{3}, \frac{2}{3},$$

etc. are exceptions. Every real number other than those numbers has a unique ternary expansion.

For example:

$$\frac{1}{3}$$

has two following expansions :

$$\frac{1}{3} = 0.1000 \ldots \text{ and } \frac{1}{3} = 0.0222 \ldots$$

$$\frac{2}{3}$$

has following two expansions :

$$\frac{2}{3} = 0.1222 \ldots \, and \, \frac{2}{3} = 0.2000 \ldots$$

The numbers $\frac{1}{2}$ *and* $\frac{1}{4}$ has expansions as :

$$\frac{1}{2} = 0.11111 \ldots.\, and\, \frac{1}{4} = 0.02020 \ldots..$$

*** CONSTRUCTION OF CANTOR SET :**

Consider the closed interval [0, 1]. The open middle part of [0, 1] is

$$\left(\frac{1}{3}, \frac{2}{3}\right)$$

Remove

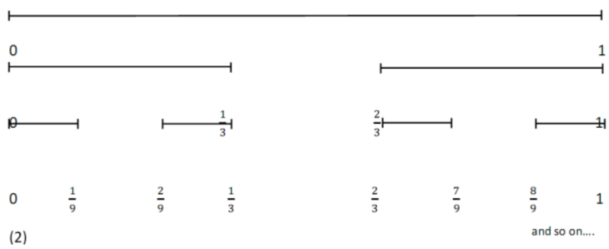
$$\left(\frac{1}{3}, \frac{2}{3}\right)$$

from [0, 1]. After removing

$$\left(\frac{1}{3}, \frac{2}{3}\right)$$

from [0, 1] we have 2 closed interval

$$\left[0, \frac{1}{3}\right] \text{ and } \left[\frac{2}{3}, 1\right]$$



In the next step; we remove the open middle third

$$\left(\frac{1}{9}, \frac{2}{9}\right) \, from \, \left[0, \frac{1}{3}\right] \, and \, \left(\frac{7}{9}, \frac{8}{9}\right) \, from \, \left[\frac{2}{3}, 1\right].$$

Now, we have the following intervals:

$$\left[0, \frac{1}{9}\right], \left[\frac{2}{9}, \frac{1}{3}\right], \left[\frac{2}{3}, \frac{7}{9}\right] \, and \, \left[\frac{8}{9}, 1\right].$$

We keep on proceeding like this. At every step, we remove the open middle third from the remaining closed interval. We proceed this process for countable number of times.

The set of points left out in this process is called the "Cantor Set".

This is also known as "Cantor ternary set" and "Cantor Middle Third Set."

NOTE 1: Any number in the open interval

$$\left(\frac{1}{3},\frac{2}{3}\right)$$

which was removed from [0, 1] in the first step has an expansion of the following form:

$$0.a_1\,a_2\,a_3\,a_4\,a_5\ldots\ldots\ldots$$

The numbers in

$$\left(\frac{1}{9},\frac{2}{9}\right),\left(\frac{7}{9},\frac{8}{9}\right),\ \ldots.$$

has ternary expression whose second place digit is 1, as we have assumed that

$$\frac{2}{3}$$

is represented as 0.01111….

Similarly, the numbers which are removed in the nth stage will have 1 in the nth position in the ternary expansion. Removing these numbers, there is no number in the Cantor Set which has digit 1 in its ternary expansion. And the numbers like

$$\frac{1}{3},\frac{2}{3}$$

etc. which were on the end points, had two expansions, from which we take the expansion without the digit 1.

NOTE 2: It appears that only the end points

$$\frac{1}{3},\frac{2}{3},\frac{1}{9},\frac{2}{9},\frac{7}{9},\frac{8}{9},\ldots\ldots$$

are remained in the Cantor Set. But, this is not true. For example: take

$$\frac{1}{4}$$

which has ternary expansion 0.0202020…. It is also in Cantor Set.

NOTE 3: We find the sum of lengths of the intervals removed from [0, 1].

$$\frac{1}{3}+\left(\frac{1}{9}+\frac{1}{9}\right)+\left(\frac{1}{27}+\frac{1}{27}+\frac{1}{27}+\frac{1}{27}\right)+\cdots=\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\cdots=\sum_{n=0}^{\infty}\frac{1}{3}\cdot\left(\frac{2}{3}\right)^n$$

$$=\frac{1}{3}\cdot\frac{1}{1-\frac{2}{3}}=1 \qquad (3)$$

*** Properties of a Cantor Set** :-

1. Cantor Set has no interior points.
2. Cantor Set is not open.
3. Cantor Set can not contain any open interval.
4. Cantor Set is bounded.
5. Cantor Set is perfect.
6. Cantor Set is closed.
7. Cantor Set has every point as its limit point.
8. Measure of Cantor Set is Zero.
9. Cantor Set is uncountable.
10. Cantor Set is compact.
11. Cantor Set has no Isolated points.

* Another Expression for Cantor Set:

$$K=\left\{x\,\middle|\,x=\sum_{n=1}^{\infty}\frac{a_n}{3^n},a_n=0\ or\ 2\right\}$$

REFERENCES:

S. C. Malik, Savita Arora (2012). ***Mathematical Analysis***: ***Real Analysis***, New Age International(P) Limited Publishers, New Delhi.

Sadhan Kumar Mapa (2015). ***Introduction to Real Analysis***, Sarat Book Distributors, Kolkata.

A.P. Singh: ***Real Analysis***, Info study Publications.

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