*Journal of Advances and Scholarly Researches in Allied Education*
*Vol. XV, Issue No. 6, August-2018, ISSN 2230-7540*

# Numerical Modelling of Application of Higher Order Accurate and Compact Numerical Scheme

**Aditya Robin Singh***

Assistant Professor, Yaduvanshi Degree College, Mahendergarh

***Abstract – Differential equations (PDE/ODEs) form the basis of many mathematical models of physical, chemical and biological phenomena, and more recently their use has spread into economics, financial forecasting, image processing and other fields. It is not easy to get analytical solution treatment of these equations, so, to investigate the predictions of PDE models of such phenomena it is often necessary to approximate their solution numerically.***

***In most cases, the approximate solution is represented by functional values at certain discrete points (grid points or mesh points). There seems a bridge between the derivatives in the PDE and the functional values at the grid points. The numerical technique is such a bridge, and the corresponding approximate solution is termed the numerical solution.***

***Keywords: Spline, Differential, Equation***

---------------------------------X---------------------------------

## INTRODUCTION

The term 'spline' is derived from the flexible device used by shipbuilder &draftsmen to draw a curve through pre-assigned points (knots) in such a way that not only the curve is continuous but also its slope and curvature are continuous functions .Draftsman attach the wooden or metal strip with weights called ducks, which can be adjusted to keep the strip in required shape. So weights are attached with the strip to keep it in the required shape.

In order to resolve the problem of working with higher degree polynomials the idea of piecewise polynomial come into existence .Instead of using polynomial for the entire domain ,the function can be approximated by several polynomials defined over the sub-domains. A polynomial which is presented over a certain domain by means of several polynomials defined over its sub-domains called a piecewise polynomial. The piecewise polynomial approximation allows us to construct highly accurate approximations, but because some approximation functions are not smooth at the point connecting separate piecewise polynomial approximation. Sometimes, while the polynomial is continuous, it may not be continuously differentiable on the interval of approximation and the graph of the interpolant may not be smooth. Splines are an attempt to solve this problem.

The underlying core of the Spline is its basis function. The defining feature of the basis function is *k* not sequence i x . Let X be a set of N+1 non decreasing real numbers. N N x ≤ x ≤ x ≤ ≤ x ≤ x 0 1 2 −1 ... .Here x s i ' are called knots , the set X is the knot sequence which represents the active area of real numbers line that defines the spline basis ,and the half –open interval[ ) 1 , i i+ x x the th i knot span. If the knots are equally spaced .,.( ) i 1 i ei x − x + is a constant for 0 ≤ i ≤ N − ),1 the knots vectors or the knot sequence is said to be uniform; otherwise, it is called non-uniform. Each spline function of degree k covers k +1 knots or k intervals.

Spline methods are a high-performance alternative to solve partial differential equations (PDEs). This paper gives an overview on the principles of Spline methodology, shows their use and analyzes their performance in application examples, and discusses its merits. Tensors preserve the dimensional structure of a discretized PDE, which makes it possible to develop highly efficient computational solvers.

## NUMERICAL MODELLING OF APPLICATION OF HIGHER ORDER ACCURATE AND COMPACT NUMERICAL SCHEME

In this study, we consider the Spline method concerning the partial differential equations:

$$\begin{cases} i\partial_t u = \sqrt{1-\Delta}u + F(u) \text{ in } \mathbb{R}^n \times \mathbb{R},\ n \geq 3, \\ u(0) = \varphi, \end{cases} \quad (1)$$

$$\begin{cases} \partial_t^2 u + (1-\Delta)u = F(u) \text{ in } \mathbb{R}^n \times \mathbb{R},\ n \geq 3, \\ u(0) = \varphi_1, \quad \partial_t u(0) = \varphi_2. \end{cases} \quad (2)$$

The nonlinear part F(u) is of Spline type such that $F(u) = V_\gamma(u)u$, where

$$V_\gamma(u)(x) = \lambda(|\cdot|^{-\gamma} * |u|^2)(x) = \lambda \int_{\mathbb{R}^n} \frac{|u(y)|^2}{|x-y|^\gamma} dy.$$

Here A is a non-zero real number and $\gamma$ is a positive number less than the space dimension n.

The equations **(4.1)** and (2) can be rewritten in the form of the integral equations

$$u(t) = U(t)\varphi - i\int_0^t U(t-t')F(u)(t')dt', \quad (3)$$

$$u(t) = (\cos t\omega)\varphi_1 + \omega^{-1}(\sin t\omega)\varphi_2 - \int_0^t \omega^{-1}(\sin(t-t')\omega)F(u)\,dt', \quad (4)$$

where $\omega = \sqrt{1-\Delta}$ and the associated unitary group U(t) is realized by the transform as

$$U(t)\varphi = (e^{-it\omega}\varphi)(x) \equiv \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}e^{-it\sqrt{1+|\xi|^2}}\widehat{\varphi}(\xi)\,d\xi,$$

where $\widehat{g}$ denotes the Fourier transform of g defined by $\widehat{g}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}g(x)\,dx.$

The operators cos $t\omega$ and sin $t\omega$ are defined by replacing $e^{-it\sqrt{1+|\xi|^2}}$ with $\cos(t\sqrt{1+|\xi|^2})$ and $\sin(t\sqrt{1+|\xi|^2})$, respectively.

If the solution u of (1) or (3) has a decay at infinity and smoothness, it satisfies two conservation laws:

$$\|u(t)\|_{L^2} = \|\varphi\|_{L^2},$$
$$E_1(u) \equiv K_1(u) + V(u) = E_1(\varphi),$$
$$K(u) = \frac{1}{2}\langle\sqrt{1-\Delta}\,u, u\rangle, V(u) = \frac{1}{4}\langle F(u), u\rangle, \quad (5)$$

Where $\langle,\rangle$ is the complex inner product in $L^2$. Also the solution of (2) or (4) or satisfies the conservation law:

$$E_2(u,\partial_t u) \equiv K_2(u,\partial_t u) + V(u) = E_2(\varphi_1,\varphi_2),$$
$$K_2(u,\partial_t u) = \frac{1}{2}(\langle\partial_t u, \partial_t u\rangle + \langle\sqrt{1-\Delta}u, \sqrt{1-\Delta}u\rangle). \quad (6)$$

The main concern of this study is to establish the global well-posedness and scattering of radial solutions of the equations (1) and (2).

The study of the global well-posedness (GWP) and scattering for the semi- relativistic equation (1) has not been long before. In (E. Lenzmann) GWP was considered with a three dimensional Coulomb type potential which corresponds to $\gamma = 1$. The first and second authors of the present study showed GWP for $0 < \gamma \leq 1$ if $n \geq 2$ and $0 < \gamma < 1$ if n=1, for $0 < \gamma < \frac{2n}{n+1}$ if $n \geq 2$, and small data scattering for $\gamma > 2$ if $n \geq 3$ In this study we tried to fill the gap $1 < \gamma \leq 2$ for GWP under the assumption of radial symmetry. For further study like blowup of solutions, solitary waves, mean field limit problem for semi-relativistic equation, see the references.

The first result is on the GWP for radial solutions of (3).

**Theorem 1.** Let **1** $< \gamma < \frac{3}{2}$ for n = **3** and $1 < \gamma < 2$ for $n \geq 4$. Let $\varphi \in H^{\frac{1}{2}}$ be radially symmetric and assume that $\|\varphi\|_{L^2}$ is sufficiently small if $\lambda < 0$. Then there exists a unique radial solution $u \in C_b H^{\frac{1}{2}}$ such that $|x|^{-1}u \in L^2_{loc}L^2$ of (3) satisfying the energy and $L^2$ conservations (5).

We mean $H_2^s$ by $H^s$ and $\dot{H}_2^s$ by $\dot{H}^s$. Hereafter, the space $L_T^q(B)$ denotes $L^q(-T,T;B)$ for $T>0$ and $\|\cdot\|_{L_T^q B}$ its norm for some Banach space B. If $T=\infty$, we use $L^q(B)$ for $L^q(\mathbb{R};B)$ with norm $\|\cdot\|_{L^q B}, 1 \leq q \leq \infty$. We also denote $v \in L_T^q(B)$ for all $T < \infty$ by $v \in L^q_{loc}(B)$

The next result is on the small data scattering of radial solutions of (4.3) for $n \geq 4$

**Theorem 2.** Let $\frac{3}{2} < \gamma < 2$ for n = **3** and $\frac{3}{2} < \gamma \leq 2$ for $n \geq 4$. Then there is a real number $s$ and $\varepsilon$ such that

$$\frac{1}{2} < s < \frac{\gamma}{2}, \quad 0 < \varepsilon < \min\left(\frac{\gamma}{2} - s, s - \frac{1}{2}\right), \quad 1 + s - \varepsilon < \gamma < 1 + s + \varepsilon. \quad (7)$$

For fixed such $s$ and $\varepsilon$, let $(\varphi_1, \varphi_2) \in D_{s+\varepsilon,s+\varepsilon} \times D_{s+\varepsilon-1,s+\varepsilon}$ be radially symmetric

data. Then if $\|\varphi_1\|_{D_{s+\varepsilon,s+\varepsilon}} + \|\varphi_2\|_{D_{s+\varepsilon-1,s+\varepsilon}}$ is sufficiently small, then there exists a unique radial solution $u \in C_b H^{s-\frac{1}{2}+\varepsilon} \cap L^2 W_{s,\varepsilon}$ to (4.4). Moreover, there exist radial functions $\varphi_1^\pm \in H^{s-\frac{1}{2}+\varepsilon}$ and $\varphi_2^\pm \in H^{s-\frac{3}{2}+\varepsilon}$ such that

$$\|u(t) - u^\pm(t)\|_{H^{s-\frac{1}{2}+\varepsilon}} \to 0 \quad \text{as} \quad t \to \pm\infty,$$

Where $u^\pm$ is the solutions to the Cauchy problem

$$\begin{cases} \partial_t^2 u^\pm + (1-\Delta)u^\pm = 0, \\ u^\pm(0) = \varphi_1^\pm, \partial_t u^\pm(0) = \varphi_2^\pm. \end{cases} \quad (8)$$

In the definit ion of initial data space $D_{\alpha,\beta}$ the space $L^{\frac{2n}{n+2-2\beta}}$ can be slightly weakened by the homogeneous Sobolev space $\dot{H}^{-(1-\beta)}$. In fact, $L^{\frac{2n}{n+2-2\beta}} \hookrightarrow \dot{H}^{-(1-\beta)}$ for $0 < \beta < 1$. Let $\tilde{D}_{\alpha,\beta}$ be the weak ened space $H^{\alpha-\frac{1}{2}} \cap \dot{H}^{-(1-\beta)}$. Then one can easily show that the solution $u \in C_b(\mathbb{R}; \dot{H}^{-(1-(s-\varepsilon))})$ and then the existence of scattering operator of (2) on a small neighborhood of the origin in $\tilde{D}_{s+\varepsilon,s-\varepsilon} \times \tilde{D}_{s+\varepsilon-1,s-\varepsilon}$. For details see Remark M below.

The lower bound $\frac{3}{2}$ of $\gamma$ is caused by the condition **(J?J)** which follows from the relation between the weight $|x|^{-a}$ and the $L^2$ estimate of Bessel function such that

$$\int_0^\infty r^{1-2a} |J_{\frac{n-2}{2}}(r)|^2 \, dr < \infty.$$

For the finiteness, the assumption $\frac{1}{2} < a < \frac{n}{2}$ is inevitable because $J_{\frac{n-2}{2}}(r) = O(r^{\frac{n-2}{2}})$ as $r \to 0$ and $J_{\frac{n-2}{2}}(r) = O(r^{-\frac{1}{2}})$ as $r \to \infty$.

For more explicit formula, see the identity below. Hence for the present it seems hard to improve the range of $\gamma$ for the small data scattering. From the perspective of negative result for the scattering[1], it will be very interesting to show the scattering up to the value of $\gamma$ greater than 1.

## CONCLUSION

In the last few years another numerical technique has been increasingly used to solve mathematical models in engineering research, the spline Method. The spline Method has a few distinct advantages over the Finite Element and Finite Difference Methods. The advantage over the Finite Difference Method is that the spline Collocation Method provides a piecewise-continuous, closed form solution. An advantage over the Finite Element Method is that the spline collocation method procedure is simpler and easy to apply many problems involving differential equations.

Our experimental results nicely confirm the excellent numerical approximation properties of Spline and their unique combination of high computational efficiency and low memory consumption, thereby showing huge improvements over standard finite-element methods.

## REFERENCES

1. Podlubny, I. (2015). Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications. Mathematics in Science and Engineering, Vol. 198. Academic Press, San Diego.

2. Miller, K.S. & Ross, B. (2013). An Introduction to the Fractional Calculus and Fractional Differential Equations.

3. Wang, Q. (2016). Numerical solutions for fractional KdV–Burgers equation by Adomian decomposition method. Appl. Math. Comput. 182(2), 1048–1055.

4. Zurigat, M., Momani, S., Alawneh, A. (2010). Analytical approximate solutions of systems of fractional algebraic–differential equations by homotopy analysis method. Comput. Math. Appl. 59(3), pp. 1227–1235.

5. Turut, V. & Güzel, N. (2013). Multivariate Pade approximation for solving nonlinear partial differential equations of fractional order. Abstr. Appl. Anal. 2013, Article ID 746401.

6. Liu, J. & Hou, G. (2011). Numerical solutions of the space-and time-fractional coupled Burgers equations by generalized differential transform method. Appl. Math. Comput. 217(16), pp. 7001–7008.

7. Khan, N.A., Khan, N.-U., Ayaz, M., Mahmood, A., Fatima, N. (2015). Numerical study of time-fractional fourth-order differential equations with variable coefficients. J. King Saud Univ., Sci. 23(1), pp. 91–98.

8. Abbas, M., Majid, A.A., Ismail, A.I.M., Rashid, A. (2014). The application of cubic trigonometric B-spline to the numerical solution of the hyperbolic problems. Appl. Math. Comput. 239, pp. 74–88.

9. Javidi, M., Ahmad, B. (2015). Numerical solution of fourth-order time-fractional partial differential equations with variable coefficients. J. Appl. Anal. Comput. 5(1), pp.52–63.

10. Kanth, A. R. & Aruna, K. (2015). Solution of fractional third-order dispersive partial differential equations. Egypt. J. Basic Appl. Sci. 2(3), pp. 190–199.

---

**Corresponding Author**

**Aditya Robin Singh***

Assistant Professor, Yaduvanshi Degree College, Mahendergarh