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A General Study on Differential and Integral Inequalities

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Abstract – The differential and integral inequalities form a subtopic of the mathematical inequalities and are discussed at length by several authors all over the world. The differential inequalities are the integration of the differential inequalities which may be called the general comparison principle. An integral inequality is a solution to the related integral inequality. By both, we mean a process of finding the bounds on the solution set of the related differential or integral inequalities. In this study, we have discussed abstract analysis, derivatives and differential inequalities, integration and integral inequalities, nonlinear analysis.

Keywords – Differential, Integral, Inequalities, Analysis, Non-Linear

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INTRODUCTION

The dominating branch of mathematics has been analysis and inequalities constitute a very significant component of the analysis. In the area of mathematics, the significance of inequalities has long been acknowledged. In the 18th and 19th centuries, renowned mathematicians laid the mathematical foundations of the theory of inequality. In the years that followed, the impact of inequalities was enormous, and many prominent mathematicians including H were drawn to the topic. However, in the 20th century, it was the pioneering work of G H. Hardy, J. E. Littlewoods, and G. Polya, which was published in 1934, that led to the subject's growth as a branch in contemporary mathematics. Then, after years of significant and productive study has been conducted on the subject of inequalities and their use in many areas of math. Despite the long history of this study, the topic of disparities is always interesting and remains a very dynamic difficult field. Indeed, it is reasonable to give the users rather than the creators of inequality the primary driving force for the fast growth of this area. It has a magnificent theory of inequality, a strong technical approach, and a very deep effect on science and technology.

It is recognized to be extremely complicated and not straightforward in most of the dynamic systems in the cosmos. These dynamic systems are therefore controlled by non-linear differential and comprehensive equations. The study of these nicht linear differential equations is the core or major portion of the nonlinear analysis. The subject of non-linear differential and integrated equations is amongst the major mathematical problems in contemporary science and technology and the theory of such equations has called most of the world's leading mathematicians into question. Even the theory of inequalities does not escape the realm of differential and integral equations. In mathematical literature, the theory of difference and integrated inequalities has long been established and these inequalities are used again as an instrument for dealing with many qualitative issues in the solutions of the corresponding differing and integral equations. There are various methods to address nonlinear differential and integrated equations such as the approximation method and theoretical methodology for the operator, etc. A nonlinear equation is normally approached by a linear equation in approximation techniques, which may be explicitly solved using the known method, and the solution of the linear equation is used as an approximation to the nonlinear differential solution. However, it must be mentioned that it is not always possible to approximate all non-linear differential equations with linear differential equations and therefore this procedure has certain restrictions. The creation of an abstract space sub-set, mapping into that by a non-linear operator expressing the differential equation, is a very necessary method for another operator, such as fixed point theorems. This needs the skills of a person who is not always feasible in the area of nonlinear functional analysis. Therefore, the strong tools to analyze non-linear differential and integral equations for various features of the solutions are differential and integral equality.

ABSTRACT ANALYSIS

Let X denotes a non-empty set and let T denote a field which may be real or complex. Then the triplet (X; +, •) is a vector or linear space, where + : X x X —> X and • : T x X -» X are two binary operations called addition and scaler multiplication in X such that,

1. (X, +) is an abelian or commutative group.

2. [(a • β) . γx] = [a • (β. γx)]. (Associativity)

3. {a + β)x = ax + β and a{x + y) = ax + ay. (Distributive laws)

4. 1 • x = x.

A linear space X together with a norm || • ||: X -> R+ becomes a normed linear space and if it is complete w.r.t. the metric defined through the norm || • ||, then X becomes a complete normed linear or Banach space. The triplet (R, +, •) is a linear space with the real field R which is further real Banach space together with a norm || • || defined as an absolute value function. Similarly, the function space C(J, R) of continuous real-valued functions is defined on a closed and bounded interval J in R. Furthermore C(J, R) is a real Banach space w.r.t. the standard supremum norm ||. || defined by ||x|| = Supt ε J |x (t)|.

A mapping T: X - X is called linear if

1. $T(x+y) = T(x) + T(y)$ for all $x, y \in X$; and

2. $T(\lambda x) = \lambda T(x)$ for all $\lambda \in \Gamma$ and $x \in X$.

$$L = \frac{d^n}{dt^n} + \frac{d^{n-1}}{dt^{n-1}} + \cdots + \frac{d}{dt}$$

$$Tx(t) = \int_a^t x(s)\,ds$$

Is a linear operator called the linear integral operator of the functions of C(J, R)? The details of linear operators appear in Lusternik and Sobolev and Kregzig.

An operator T: X —> X is nonlinear if it is not linear. The theory of nonlinear operators is well developed and there are plenty in numbers. The details are given in Granas and Dugundji and Heikkila and Lakshmikantham.

Similarly, a function f (t, x) defined on J x R into R is called linear in x if the map x -f(t, x) satisfies two conditions of linearity in the variable x for each t G J. The function (£, x) = ax is linear in x. Again, a function f (t, x) defined on J x 1 into R is called nonlinear in x if it is not linear for each t e J. There are ample examples of nonlinear functions available in the literature. A simple example is (£, x) = t + x which is nonlinear in x. An equation involving a nonlinear function or nonlinearity is called a nonlinear functional equation.

DERIVATIVES AND DIFFERENTIAL INEQUALITIES

Certain basic comparison results, that concerned with estimating a function that satisfies a differential inequality by the external solutions of the corresponding differential equation. Sometimes, it is enough to have the differential inequality satisfied relative to only Dini derivatives. We adopt the following notation for Dini derivatives:

$$D^+u(t) = \limsup_{h \to 0^+} \frac{1}{h}[u(t+h) - u(t)];$$

$$D_+u(t) = \liminf_{h \to 0^+} \frac{1}{h}[u(t+h) - u(t)];$$

$$D^-u(t) = \limsup_{h \to 0^-} \frac{1}{h}[u(t+h) - u(t)];$$

$$D_-u(t) = \liminf_{h \to 0^-} \frac{1}{h}[u(t+h) - u(t)],$$

Where,

$u \in C((t, t_0 + a), \mathbb{R})$. When $D^+u(t) = D_+u(t)$, U'+ denotes the right derivative (t). Such a link is also marked by u' (t when D u(t) = _u. (t).

Lemma 1.1 Suppose m (t) is continuous on-(a, b). Then m(t) is no decreasing (no increasing) on (a, b) if and only if D+m(t) > 0(< 0) for every t G (a, b), where

$$D^+m(t) = \limsup_{\delta \to 0^+} \frac{1}{\delta}[m(t+\delta) - m(t)]$$

PROOF. The condition is necessary. Let us prove that it is sufficient. Assume first that D+m(t) > 0 on (a, b).

$$\frac{m(t) - m(\zeta)}{t - \zeta} < 0$$

Which implies D+m(C) < 0. This is a contradiction and therefore the proof is complete.

Lemma 1.2 Let v,w ε C([to,T],R) For certain fixed derivatives of Dini,

$$Dv(t) \le w(t),\ t \in [t_0, T].$$

PROOF. Define the function

$$m(t) = v(t) - \int_{t_0}^{t} w(s)ds$$

It then follows, from the assumption, that

$$Dm(t) = Dv(t) - w(t) \leq 0, \quad t \in [t_0, T]$$

Thus, m(t) does not rise int [to, T] by Lemma. Consequently,

$$D_-m(t) = D_-v(t) - w(t) \leq 0, \quad t \in [t_0, T],$$

And the lemma is proved.

$$\text{If } D^+u(t) = D_+u(t) = D^-u(t) = D_-u(t),$$

then the common value is called the derivative of the function u and we denote it by du/dt, and the function u is called differentiable or derivable on J. The following result is sometimes useful while establishing the nonlinear Lipschitz condition of a differentiable function on the domain of its definition.

Theorem 1.1 If the function f defined during a shutdown period is actually evaluated [a, b]

(a) Continuous on [a, b], and

(b) Derivable on the open interval (a, b),

Then Here exists at least a point c between a and b (c ε (a, b)) such that

$$f(b) - f(a) = f'(c)(b - a)$$

The existence theorem together with the extension result implies the following.

Theorem 1.2 Let g ε C (E, R), where E = J x R is an open (t, u)-set in R2 and (to, uo) ε E. Then, the IVP

$$u' = g(t, u), \quad u(t_0) = u_0,$$

Have external solutions (that is, minimal and maximal solutions) which can be extended up to the boundary of E. The next lemma is useful in certain situations.

INTEGRATION AND INTEGRAL INEQUALITIES

Let J = [a, b] be a closed and bounded interval in R, the set of real numbers and let x: J —> R be a bounded function. Let P = {x0,x1,….,xn } be a partition of J, where a = x0<x1<….<xn=b.Denote by Sn, the sum

$$S_n = \sum_{i=1}^{n} x_i \Delta_i$$

Where $x_i \in [x_{i-1}, x_i]$ and $\Delta_i = x_i - x_{i-1}$, the length of the ith subinterval of the partition p. Then the Riemann integral of the function x over J is defined as

$$\lim_{n \to \infty} S_n = \int_a^b x(s)\,ds = \int_J x(s)\,ds$$

Provided the limit exists. If the above limit exists, we say x is Riemann's integrable function on J. The Riemann integration enjoys the following properties.

If x and y are two Riemann integrable functions defined on J, then (a) x±.y is again an integrable function with

$$\int_a^b (x \pm y)(s)\,ds = \int_a^b x(s)\,ds \pm \int_a^b y(s)\,ds.$$

(b) $\int_a^b \lambda x(s)\,ds = \lambda \int_a^b x(s)\,ds$ *for all* $\lambda \in \mathbb{R}$.

(c) *If* $0 \leq x(s) \leq y(s)$ *for each* $s \in J$, *then*

$$0 \leq \int_a^b x(s)\,ds \leq \int_a^b y(s)\,ds.$$

(d) $\left| \int_a^b x(s)\,ds \right| \leq \int_a^b |x(s)|\,ds.$

The following theorem is oftentimes used in the theory of initial value problems of ordinary nonlinear differential equations.

Theorem 1.3 A function f is bounded and integrable on [a, b], and there exists a function F such that F' = f on [a, b], then

$$\int_a^b f(s)\,ds = F(b) - F(a).$$

There are two types of integral inequalities that we come across in the literature. »The integral inequalities of the first type are similar to differential inequalities and obtained exactly in the same way. Such integral inequalities may be called comparison theorems for integral equations. The second type of integral inequalities is in terms of the bound on the unknown function appearing under the integral sign and not in terms of the maximal solution existing on the domain of definition of integration. These integral inequalities are employed for proving the stability of certain nonlinear differential and integral equations. Below we state some basic Gronwail and Bellman

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integral inequalities of the second type which have been widely used or find numerous applications in the study of various classes of nonlinear differential and integral equations.

Theorem 1.4 Let u be a continuous function defined on the closed interval [a, a+ h] and

$$0 \le u(t) \le \int_{\alpha}^{t} [bu(s) + a]\, ds, \ \ t \in J$$

Where a and b are nonnegative constants. Then

$$0 \le u(t) \le ahe^{bh}, \ t \in J.$$

Note that Peano (1886-96) deals with the specific situation with = 0 and demonstrated some very general findings of differential inequalities as well as maximum and minimum solutions for differential equations. The following total inequality was established by Bellman in 1943.

Theorem 1.5 Let u and f be two continuous and nonnegative real-valued functions defined on J = [t0, t0 + a] and let c be a nonnegative constant. Then the inequality,

$$u(t) \le c + \int_{t_0}^{t} f(s)u(s)\, ds, \ \ t \in J$$

Implies that

$$u(t) \le c \exp\left(\int_{t_0}^{t} f(s)\, ds\right) \ \ t \in J.$$

Bellman's finding incorporates that of Gronwall's Theorem because $\int_{t_0}^{t} a\, ds \le a\alpha$ for $t_0 \le t \le t_0 + h.$ Bellman's integral inequality given in above in later years, Theorem 1.4.2 had an enormous impact and the study of such disparities became a significant topic. The integral inequalities of the above type are called Gronwall-Bellman inequalities which are further generalized by several authors in various directions. Therefore, Gronwall-Bellman inequalities are basic and responsible for the multitude of development of the topic of integral inequalities and applications. For the analysis of the asymptotic behavior of linear differential equations solutions, Bellman (1958) showed that, in theorem 1.4.3, integrals were the integral inequalities he created himself.

Theorem 1.6 Let us make u and f be both permanent and nonnegative real-valued functions defined by J=[to,to+a] and let n(t), defined by J= [to,to+a], and let n(t). Then the inequality

$$u(t) \le n(t) + \int_{t_0}^{t} f(s)u(s)\, ds, \ \ t \in J$$

Implies that

$$u(t) \le n(t) \exp\left(\int_{t_0}^{t} f(s)\, ds\right) \ \ t \in J.$$

Gollwitzer gave the following generalization of GronwailBellmann's inequality.

Theorem 1.7 Allow u, f, g, and h to be defined non-negative continually on J = [t0, t0 = a] and

$$u(t) \le f(t) + g(t) \int_{t_0}^{t} h(s)u(s)\, ds, \ \ t \in J.$$

Then

$$u(t) \le f(t) + g(t) \int_{t_0}^{t} h(s)f(s)\exp\left(\int_{s}^{t} h(\tau)g(\tau|\, d\tau\right) ds, \ t \in J.$$

In establishing several extensions of Bellman's inequality Pachpatte has proven the following version of integral inequality provided in Theorem above.

Theorem 1.8 let u, g, and h be nonnegative continuous junctions defined on J = [to, to = a], n(t) be a continuous, positive, and no decreasing function defined on J and

$$u(t) \le n(t) + g(t) \int_{t_0}^{t} h(s)u(s)\, ds, \ \ t \in J.$$

Then

$$u(t) \le n(t) \left[1 + g(t) \int_{t_0}^{t} h(s)\exp\left(\int_{s}^{t} h(\tau)g(\tau|\, d\tau\right) ds\right], \ t \in J.$$

We will next use the integral inequality theoretically stated. Consider the non-linear integrated quadratic equation

$$x(t) = k(t, x(t)) + [f(t, x(t))] \left(\int_{0}^{t} g(s, x(s))\, ds\right)$$

For all t ε J = [0, T], where k, f, g: J x R — R are continuous real-valued functions.

The above integral equation has been considered for the study by several authors in the literature for different aspects of the solutions. See Dhage and the references therein. We establish the bounded of the solutions of the above quadratic integral equation 1.4*12 by an application of the integral equality established in Theorem under suitable conditions.

We assume the following hypothesis in what follows.

(H1) There exist continuous nonnegative and no decreasing functions o, b, c: J —> R such that

$$|k(t, x(t))| \le a(t),$$
$$|f(t, x(t))| \le b(t)$$

And

$$|g(t, x(t))| \le c(t)|x(t)|$$

For all t ε [0, T].

Theorem 1.9 Assume that Hypothesis (H1) holds. Then every solution of the quadratic integral equation defined on J is bounded.

PROOF. The given integral equation is

$$x(t) = k(t, x(t)) + [f(t, x(t))]\left(\int_0^t g(s, x(s))\,ds\right)$$

Which further gives,

$$\begin{aligned} |x(t)| &\le |k(t, x(t))| + |f(t, x(t))| \left|\int_0^t g(s, x(s))\,ds\right| \\ &\le |k(t, x(t))| + |f(t, x(t))| \int_0^t |g(s, x(s))|\,ds \\ &\le a(t) + b(t)\int_0^t c(s)|x(s)|\,ds. \end{aligned}$$

Now we apply a variant of Bellman's integral inequality due to Paehpatte formulated in Theorem, we obtain

$$\begin{aligned} |x(t)| &\le a(t)\left[1 + b(t)\int_0^t c(s)\exp\left(\int_s^t c(\tau)b(\tau)\,d\tau\right) ds\right] \\ &= M(t) \end{aligned}$$

For every t µ J in which M(t) is an ongoing function in J. Because every continuous function reaches a maximum in M at a closed and bounded interval. Therefore, the number M* = sup M (t) t ε J exists, hence |x(t)| < M* for all t ε J. As a result, every solution of the given integral equation is bounded on J.

NONLINEAR ANALYSIS

Fixed item theory is an essential subject in nonlinear functional analysis that offers some strong tools for nonlinear equations research. Banach (see Agarwal et.al.), Granas and Dugundji, and Krasnoselskii have started the systematic study of the non-linear equations using the fixed point theory. For some of the initial values issues of nonlinear ordinary differential and integrated differential equations, we utilized certain known and newly-established fixed point theorems for non-lingerie operators in this research.

Definition 1: Let X e an abstract space and let T: X —> X be an operator. A variable £ ε X is called a fixed point of the operator T if T ε = ε and A fixed point theorem is referred to as any mathematical declaration affirming the presence of a fixed point.

There are only basic three approaches for dealing with the nonlinear differential and integral equations, namely,

(1) Geometric methods,

(2) Topological methods, and

(3) Algebraic methods.

Geometric fixed point theory: The following is similar to the classical or determinist contractions mapping idea, the only geometrical theorem with a high profile that is frequently employed in the study of nonlinear differential and integer equations.

Theorem 1.10 Let (E,d) be a complete metric space and T: E ↔ E is a contraction mapping, that is, for every x,y ε E,

$$d(Tx, Ty) \le \alpha d(x, y), \quad 0 < \alpha < 1.$$

Then, there exists a unique faced point x of T in E such that x = Tx.

Using the above-mentioned fixed point theorem not only creates a unique existence but also solutions for non-linear differential and integrated equations. The aforementioned technique should be used only if specific Lipschitz conditions are satisfied by the nonlinearity involved in the nonlinear equations. The Banach contraction concept gives the conventional iteration or following methods an abstract framework.

Topological fixed point theory: The topological fixed o point theory for completely continuous operators is generally used in the study of solutions for differential and integral equations. We recall a few definitions from Banach spaces.

Let T: X —> X be a operator. T is called compact if T(X) is a compact subset of X. T is termed completely limited when T(B) is a completely limited X-sub-set, where B is a limited X-set. Finally, when the operator is continuous and fully limited by x, T is termed continuous. Note that any compact operator on X is completely bound, while the opposite is not always true, although both concepts correspond with the bounded X sub-sets. The applicable fixed-point theorem is,

Theorem 1.11 Let S be a closed, convex, and bounded subset of the Banach space X and let T: X —> X be an operator. Then T has a fixed point.

Schauder's fixed point theorem, which we will present next, is another fixed point finding with numerous applications in the theory of functional equations.

Theorem 1.12 Let E be a Banach space and B ⊂ E, a convex, closed bounded set. IfT:E—>E is a constant user so that T{B) to B and T is rather compact, T has a fixed point.

In a Frechet space setting, one more general conclusion known as the fixed-point theorem of TychonofF is. Let us define Frechet space before stating the theorem. A Frechet space is a linear space provided with an invariant translation metric. Using seminorms, Frechet's space is defined. A linear space map X on a semi normal is the map of X to R+, say x —> |x|, so that:

(i) $|x| \geq 0$;

(ii) $|\lambda x| = |\lambda||x|$;

(iii) $|x + y| \leq |x| + |y|$.

The main difference from a standard is that |x| = 0 doesn't always mean x = 0.

Theorem 1.13 Let F be a Frechet space whose distance function is constructed utilizing a sufficient, countable family of seminorms, say $\{|x_k|; k \geq 1\}$,i.e., from $|x|_k = 0,\ k \geq 1$ one derives x = 0. If B ⊂ F is a closed, convex set and T: B —» B is a continuous operator 'such that TB is relatively compact, then T has at least one fixed point in B.

The above fixed-point theorems have several versions useful in applications. The topological methods are purely existential and only applicable if the nonlinearity involved in the nonlinear equations satisfy certain compactness type conditions.

Algebraic fixed-point theory: The algebraic fixed point theory for monotone and continuous operators is generally useful in the study of solutions for differential and integral equations. We recall a few definitions concerning the monotone operators in Banach spaces.

Let X be an ordered Banach space with an order relation < induced by the order cone K in X. Let a and b be two elements in the Banach space X such that a < b. Then we define an order interval [a, b] in X by

$$[a, b] = \{x \in X \mid a \leq x \leq b\}.$$

Definition 2: A operator Q on an ordered Banach space X into itself is called nondecreasing if for any x, ε X, x < y implies Qx < Qy.

Theorem 1.14 Let a and b be two elements of X such that a < b and let [a, b] be an order interval in the ordered Banach space X. If Q: [a, b] —» [a, b] Be a fully continuous and non-diminished operator and the K cone is normal for the X and then Q has a minimum fixed point of £* and a larger fixed point of £* for [a, b]. Moreover,

$$\xi_* = \lim_{n\to\infty} Q^n a \quad and \quad \xi^* = \lim_{n\to\infty} Q^n b.$$

The algebraic fixed-point method yields the theoretical successive approximations. results for the existence of solutions. It also gives qualitative information about the solutions for the equations. These methods are applicable only if the nonlinearity involved in the differential and integral equations are monotonic and satisfy certain compactness type conditions.

CONCLUSION

In this study, we have discussed abstract analysis, derivatives and differential inequalities, integration and integral inequalities, nonlinear analysis. Which concludes in integral inequalities that the number M* = sup M (t) t ε J exists, hence |x(t)| < M* for all t ε J. As a result, every solution of the given integral equation is bounded on J. In non-linear analysis, the algebraic fixed-point method yields the theoretical successive approximations. results for the existence of solutions. It also gives qualitative information about the solutions for the equations.

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