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A Study of Efficient Numerical Methods and their Applications in Fractional Differential Equations

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Abstract - Fractional calculus is the study of unique fractional orders-integral operators with useful applications in a number of industries of engineering and research. A fractional differentiation is simply an operator who takes a broad perspective of the normal differentiation. Fractional derivative equations, such as real or complex order differentiation, have not defined a vital role in producing the extraordinary complexity of various components that rely on difficult structures in some of the most diversified areas of engineering and research. Now, here we demonstrate the most important as well as useful improvements in nonlinear non-fractional derivative models mathematicians investigated as employed by authors at least to represent the dynamics of ways in atypical media. Fractional calculus, in the sense that it extends the principle of derivatives and integrals to include arbitrary order, may be seen as extension of classical calculus. Efficient math modeling by differential equation on the order of fractional necessitates the development of accurate and scalable computer methods. In this paper discuss the efficient numerical methods and their applications in fractional differential equations.

Keywords - efficient numerical methods, applications, fractional differential equations

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INTRODUCTION

Fractional differential equations can only be solved in series because of the difficulty of calculating with fractional operators. A approach that is effective for solving equations with fractional derivatives is the fractional power series method. M. Kurulay and M. Bayram provide a technique for solving fractional PDEs using a power series.

The power series form (PSM) is given as follows:

$$\sum_{n=0}^{\infty} c_n(\varepsilon-\varepsilon_0)^{n\alpha} = c_0 + c_1(\varepsilon-\varepsilon_0)^{\alpha} + c_2(\varepsilon-\varepsilon_0)^{2\alpha} + c_3(\varepsilon-\varepsilon_0)^{3\alpha} + \ldots,$$

where $0 \le m-1 < \alpha \le m$ and c_n are constants, $\varepsilon_0 = 1$ leads the series to fractional Maclaurin series. If the fractional power series $\sum_{n=0}^{\infty} c_n(\varepsilon-\varepsilon_0)^{n\alpha}$ converges whenever ε is greater than or equal to zero but less than b, where b is positive. Further if the fractional power series $\sum_{n=0}^{\infty} c_n(\varepsilon-\varepsilon_0)^{n\alpha}$ diverges when ε is greater than d, where d is positive.

3.1.1 Application of PSM to solve compound fractional relaxation equation.

Consider the following compound fractional relaxation equation as follows:

$$D^{\frac{1}{2}}u - aD^{\alpha}u - bu = 0, \qquad (\varepsilon > 0,\ 0 < \alpha \le 1), \tag{1}$$

with $u(0) = 1$

Let we take solution of (1) in the subsequent series form

$$u(\varepsilon) = \sum_{n=0}^{\infty} c_n \varepsilon^{n\alpha}. \tag{2}$$

$$\sum_{n=0}^{\infty} c_n D^{\frac{1}{2}} \varepsilon^{n\alpha} - a\sum_{n=1}^{\infty} c_n \frac{\Gamma(1+n\alpha)}{\Gamma(1+\alpha(n-1))} \varepsilon^{(n-1)\alpha} - b\sum_{n=0}^{\infty} c_n \varepsilon^{n\alpha} = 0. \tag{3}$$

By comparing coefficients of $\varepsilon^{n\alpha}$, we get recurrence relation to get all information about constants.

If we put $\alpha = \frac{1}{2}, a = b = 1,$ in eq. (1), we get

$D^2u - D^{1/2}u - u = 0$, Subjected to $u(0) = 1, u'(0) = 1$,

with solution $u(\varepsilon) = \sum_{n=0}^{\infty} c_n \varepsilon^{n/2}$.

Now, $u(0) = c_0 = 1, u'(0) = 0 \Rightarrow c_1 = c_2 = 0$.

$$\sum_{n=0}^{\infty} c_n D^2 \varepsilon^{n/2} - \sum_{n=1}^{\infty} c_n \frac{\Gamma(1+\frac{n}{2})}{\Gamma(1+\frac{n-1}{2})} \varepsilon^{\frac{n-1}{2}} - \sum_{n=0}^{\infty} c_n \varepsilon^{n/2} = 0,$$

$$\sum_{n=4}^{\infty} c_n \frac{n}{2}\left(\frac{n}{2}-1\right) \varepsilon^{\frac{n-4}{2}} - \sum_{n=1}^{\infty} c_n \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n-1}{2}+1)} \varepsilon^{\frac{n-1}{2}} - \sum_{n=0}^{\infty} c_n \varepsilon^{n/2} = 0,$$

by comparing coefficients of $\varepsilon^{n/2}$, we get

$$c_{n+4}\left(\frac{n+4}{2}\right)\left(\frac{n+4}{2}-1\right) - c_{n+1} \frac{\Gamma(\frac{n+1}{2}+1)}{\Gamma(\frac{n}{2}+1)} - c_n = 0,$$

$$c_{n+4} = \left(c_{n+1} \frac{\Gamma(\frac{n+1}{2}+1)}{\Gamma(\frac{n}{2}+1)} + c_n \right) \frac{4}{(4+n)(2+n)}, \quad (0 \le n).$$

For n = 0

$$c_4 = \frac{1}{2}\left(c_1 \frac{\Gamma(\frac{3}{2})}{\Gamma(1)} + c_0 \right) \Rightarrow c_4 = \frac{1}{2}.$$

For n = 1

$$c_5 = \frac{4}{15}\left(c_2 \frac{\Gamma 2}{\Gamma \frac{3}{2}} + c_1 \right) \Rightarrow c_5 = 0$$

On solving, we get

$$c_0 = 1, c_1 = c_2 = 0, c_4 = \frac{1}{2}0, c_5 = 0, c_6 = c_3 \frac{\sqrt{\pi}}{8}, c_7 = \frac{16}{105\sqrt{\pi}} + \frac{4}{35}c_3, c_8 = \frac{1}{24}.$$

$$u(\varepsilon) = c_0 + c_1\varepsilon^{1/2} + c_2\varepsilon + c_3\varepsilon^{3/2} + c_4\varepsilon^2 + c_5\varepsilon^{5/2} + c_6\varepsilon^3 + c_7\varepsilon^{7/2} + c_8\varepsilon^4 \ldots\ldots..,$$

$$u_8(\varepsilon) = 1 + c_3\varepsilon^{3/2} + \frac{1}{2}\varepsilon^2 + \frac{c_3}{8}\sqrt{\pi}\varepsilon^3 + \left(\frac{16}{105\sqrt{\pi}} + \frac{4}{35}c_3\right)\varepsilon^{7/2} + \frac{1}{24}\varepsilon^4.$$

The above result is 8th approximation results of (1) when $\alpha = \frac{1}{2}$. Advantage of power series method to freedom of choosing any point in the interval of integration with approximate solution.

MODIFIED VARIATION ITERNATION METHOD (MVIM)

Using Laplace transforms, Guo-Cheng Wu and Dumitru Baleanu developed a unique variational iteration approach that incorporates Lagrange multipliers. Nonlinear fractional derivative problems occur often in mathematical physics and other related fields, and Lagrange multiplier methods have been extensively employed to solve these equations. He developed the variational iteration method for solving nonlinear equations. As a result of the VIM method's adaptability and durability, it has become an increasingly popular tool for researchers to employ in a wide range of applications. Setting the correlation functional; identifying the Lagrange multipliers; setting an initial iteration are all phases in this approach. Poor convergence occurs when the approach is applied to ordinary differential equations using the Lagrange multiplier. For fractional differential equations with starting values, the modified VIM technique has solved this issue by defining a Lagrange multiplier from the Laplace transform. To further understand the MVIM approach, let's look at the following nonlinear differential problem.

$$\frac{d^m u}{d\varepsilon^m} + R_1(u) + N_1(u) = f(\varepsilon), \qquad 5$$

with the initial conditions

$$u^{(k)}(0) = u_0^k \ldots., \qquad 6$$

for k = 0,1, 2,.....,m =1,where $u = u(\varepsilon)$, R_1 is a linear and N_1 is a nonlinear bounded operator, $f(\varepsilon)$ is known as regular function .

Now take Laplace transform of equation (5), to construct correction functional is

$$U_{n+1}(s) = U_n(s) + \lambda(s)\left(s^m U_n(s) - s^{m-1}u(0) - \ldots - u^{m-1}(0) + L[R_1(u_n) + N_1(u_n) - f(\varepsilon)]\right) \qquad 7$$

Regarding the terms $L[R_1(u_n) + N_1(u_n) - f(\varepsilon)]$ as restricted variations to make equation fixed with respect to U_n.

$$\delta U_{n+1}(s) = \delta U_n(s) + \lambda(s)\left(s^m \delta U_n(s)\right) \qquad 8$$

4 From (8), we calculate Lagrange multiplier as

$$\lambda(s) = -\frac{1}{s^m}.$$

The successive approximations can be given by taking inverse Laplace transform of equation

(7)

$$u_{n+1}(\varepsilon) = u_n(\varepsilon) - L^{-1}\left\{\frac{1}{s^m}\left(s^m U_n(s) - s^{m-1}u(0) - \ldots - u^{m-1}(0) + L[R_1(u_n) + N_1(u_n) - f(\varepsilon)]\right)\right\}$$
$$= L^{-1}\left(\frac{u(0)}{s} + \ldots\ldots + \frac{u^{m-1}(0)}{s^m}\right) - L^{-1}\left[\frac{1}{s^m}\left(L[R_1(u_n) + N_1(u_n) - f(\varepsilon)]\right)\right]. \qquad 9$$

with initial conditions

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$$u_0(\varepsilon)=L^{-1}\left(\frac{u(0)}{s}+\dots+\frac{u^{m-1}(0)}{s^m}\right)=u(0)+u'(0)\varepsilon+\dots+u^{(m-1)}(0)\frac{\varepsilon^{m-1}}{(m-1)!}. \qquad 10$$

3.2.1. Application of MVIM in Solution of composite fractional oscillation equation

We now apply MVIM to solve following composite fractional equation

$$D^{\alpha_1}u=a_1D^2u-b_1u-c_1,(\varepsilon>0,\ 0<\alpha_1\le 1), \qquad 11$$

subjected to the conditions

$$u(0)=0,\ u'(0)=0.$$

According to MVIM method the correction function for equation (11) has constructed as follows:

$$u_{n+1}(s)=u_n(s)+\lambda(s)\left(s^{\alpha_1}u_n(s)-s^{\alpha_1-1}u(0)-s^{\alpha_1-2}u'(0)+L\left(b_1u_n+c_1-a_1D^2u_n\right)\right),$$

where $\lambda(s)=-\frac{1}{s^{\alpha_1}}$ & taking Inverse Laplace transform

$$u_{n+1}(\varepsilon)=L^{-1}\left[s^{-1}u(0)+s^{-2}u'(0)-\frac{1}{s^{\alpha_1}}L\left(b_1u_n+c_1-a_1D^2u_n\right)\right],$$

$$u_{n+1}(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha_1}}L\left(b_1u_n+c_1-a_1D^2u_n\right)\right].$$

Case I: When n = 0, above equation gives first iteration.

$$u_1(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha_1}}L\left(b_1u_0+c_1-a_1D^2u_0\right)\right],$$

$$u_1(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha_1}}L(c_1)\right],$$

$$u_1(\varepsilon)=-(c_1)\frac{\varepsilon^{\alpha_1}}{\Gamma(\alpha_1+1)}.$$

Case II: For n = 1, yields

$$u_2(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha_1}}L\left(b_1u_1+c_1-a_1D^2u_1\right)\right],$$

$$u_2(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha_1}}L\left(-b_1(c_1)\frac{\varepsilon^{\alpha_1}}{\Gamma(\alpha_1+1)}+c_1+\frac{a_1(c_1)}{\Gamma(\alpha_1+1)}\alpha_1(\alpha_1-1)\varepsilon^{\alpha_1-2}\right)\right],$$

$$u_2(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha_1}}\left(-b_1(c_1)\frac{1}{s^{\alpha_1+1}}+c_1\frac{1}{s}+\frac{a_1(c_1)}{\Gamma(\alpha_1+1)}\alpha_1(\alpha_1-1)\frac{\Gamma(\alpha_1-1)}{s^{\alpha_1-1}}\right)\right],$$

$$u_2(\varepsilon)=L^{-1}\left[b_1(c_1)\frac{1}{s^{2\alpha_1+1}}-c_1\frac{1}{s^{\alpha_1+1}}-\frac{a_1(c_1)}{\Gamma(\alpha_1)}\Gamma(\alpha_1)\frac{1}{s^{2\alpha_1-1}}\right],$$

$$u_2(\varepsilon)=b_1(c_1)\frac{\varepsilon^{2\alpha_1}}{\Gamma(2\alpha_1+1)}-c_1\frac{\varepsilon^{\alpha_1}}{\Gamma(\alpha_1+1)}-a_1(c_1)\frac{\varepsilon^{2\alpha_1-2}}{\Gamma(2\alpha_1-1)}. \qquad 12$$

3.2.2 Solution Of Nonlinear Fraction Equation By MVIM.

In this section, MVIM method is applied to solve following equation

$$D_0^{\alpha}y(\varepsilon)-y^2(\varepsilon)-1=0, \qquad (m-1<\alpha\le m,\ t\ge 0), \qquad 13$$

subjected to the conditions $y^{(i)}(0)=0$, (i= 0, 1, … , m – 1, m), m is positive integer.

As per MVIM method, the correction function for equation (4.13) is given as follows:

$$Y_{n+1}(s)=Y_n(s)+\lambda(s)\left(\left(s^{\alpha}Y_n(s)-s^{\alpha-1}Y(0)-s^{\alpha-2}Y'(0)\dots\dots\dots\right)+L\left(-y_n^2(\varepsilon)-1\right)\right)$$

For λ (s) = $-\frac{1}{s^{\alpha}}$ & taking inverse Laplace transform

$$y_{n+1}(\varepsilon)=L^{-1}\left[\left(s^{-1}y(0)+s^{-2}y'(0)\dots\right)-\frac{1}{s^{\alpha}}L\left(-y_n^2(\varepsilon)-1\right)\right].$$

Let we take m = 1 in (4.13), $(0<\alpha\le 1)$,

$$y_{n+1}(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha}}L\left(-y_n^2(\varepsilon)-1\right)\right].$$

On putting n = 0

$$y_1(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha}}L(-1)\right]=L^{-1}\left[\frac{1}{s^{\alpha+1}}\right]=\frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)}.$$

For n = 1

$$y_2(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha}}L\left(-y_1^2(\varepsilon)-1\right)\right],$$

$$y_2(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha}}L\left(-\frac{\varepsilon^{2\alpha}}{\Gamma(1+\alpha)^2}-1\right)\right],$$

$$y_2(\varepsilon)=L^{-1}\left[\frac{1}{s^{\alpha}}\left(\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}s^{-2\alpha-1}+\frac{1}{s}\right)\right]=L^{-1}\left[\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^2}\frac{1}{s^{3\alpha+1}}+\frac{1}{s^{\alpha+1}}\right],$$

$$y_2(\varepsilon)=\frac{\Gamma(2\alpha+1)}{(\Gamma(1+\alpha))^2}\frac{\varepsilon^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)}.$$

For n = 2

$$y_3(\varepsilon)=L^{-1}\left[-\frac{1}{s^{\alpha}}L\left(-y_2^2-1\right)\right]$$

$$=L^{-1}\left[\frac{1}{s^{\alpha}}L\left(\left(\frac{\Gamma(2\alpha+1)}{(\Gamma(1+\alpha))^2}\frac{\varepsilon^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)}\right)^2+1\right)\right]$$

$$=L^{-1}\left[\frac{1}{s^{\alpha}}L\left(\frac{(\Gamma 2\alpha+1)^2}{(\Gamma(\alpha+1))^4(\Gamma(3\alpha+1))^2}\varepsilon^{6\alpha}+\frac{\varepsilon^{2\alpha}}{(\Gamma\alpha+1)^2}+2\frac{\Gamma(2\alpha+1)}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)}\varepsilon^{4\alpha}\right)+1\right]$$

$$=L^{-1}\left[\frac{(\Gamma(2\alpha+1))^2}{(\Gamma\alpha+1)^4(\Gamma 3\alpha+1)^2}\frac{\Gamma(6\alpha+1)}{s^{7\alpha+1}}+\frac{1}{(\Gamma\alpha+1)^2}\frac{\Gamma(2\alpha+1)}{s^{3\alpha+1}}+\frac{2\Gamma(2\alpha+1)}{(\Gamma\alpha+1)^3\Gamma(3\alpha+1)}\frac{\Gamma(4\alpha+1)}{s^{5\alpha+1}}+\frac{1}{s^{\alpha+1}}\right],$$

$$y_3(\varepsilon)=\left[\frac{(\Gamma(2\alpha+1))^2\Gamma(6\alpha+1)}{(\Gamma\alpha+1)^4(\Gamma 3\alpha+1)^2}\frac{\varepsilon^{7\alpha}}{\Gamma(7\alpha+1)}+\frac{\Gamma(2\alpha+1)}{(\Gamma\alpha+1)^2}\frac{\varepsilon^{3\alpha}}{\Gamma(3\alpha+1)}+\frac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)}{(\Gamma\alpha+1)^3\Gamma(3\alpha+1)}\frac{\varepsilon^{5\alpha}}{\Gamma(5\alpha+1)}+\frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)}\right],$$

which is third approximate solution. For α = 1, in above equation yields

$$y_3(\varepsilon)=\varepsilon+\frac{\varepsilon^3}{3}+\frac{2}{15}\varepsilon^5+\dots\approx\tan\varepsilon. \qquad 14$$

Since (13) is an ordinary case problem, we may infer that the third approximation of MVIM's series solution is an exact match with the actual answer, making it obvious that MVIM's accuracy is improved by taking more components of the solution into account.

3.2.3 Solution of Fractional Sir Model By Mvim Method

VIM approach is now used to solve the fractional Susceptible-Infected-Recovered model. Developed by W.O. Kermack, the SIR model has played a significant role in mathematical epidemiology in calculating the population's prevalence of various diseases and the number of persons who have recovered.

Fractional model start with some basic notations:

S (ε) indicate quantity of susceptible people at time ε

I (ε) indicate quantity of infected people at time ε

R (ε) indicate quantity of recovered people at time ε

$$S(\varepsilon)+I(\varepsilon)+R(\varepsilon)=N,$$

where N is overall population size.

Following the assumptions, we use a fractional order SIR model.:

$$D_{\varepsilon}^{\alpha_1} x(\varepsilon)=-\beta\, x(\varepsilon)\, y(\varepsilon), \qquad 15$$

$$D_{\varepsilon}^{\alpha_2} y(\varepsilon)=\beta\, x(\varepsilon)\, y(\varepsilon)-k\, y(\varepsilon), \qquad 16$$

$$D_{\varepsilon}^{\alpha_3} z(\varepsilon)=k\, y(\varepsilon), \qquad 17$$

where $x(\varepsilon)$, $y(\varepsilon)$, $z(\varepsilon)$ represents $S(\varepsilon)$, $I(\varepsilon)$, $R(\varepsilon)$ respectively and $0 \le \alpha_i \le 1$ for i=1, 2, 3; $k \ge 0$ is the recovery rate , $\beta \ge 0$ is the average number of transmission from an infected person in a time ε.

In the fractional SIR system (equations (15) to (17)), a sophisticated mathematical technique called the modified VIM approach with opening scenarios was used to achieve an approximate analytical solution

$x(0)=\delta,\ y(0)=\gamma,\ z(0)=\mu$ . .For each fractional ordertime derivative of FSIR, numerical computations are performed, which are illustrated visually. The following is how equation (15)'s correction function was generated using the MVIM method:

$$X_{n+1}(s)=X_n(s)+\lambda(s)\left(s^{\alpha_1}X_n(s)-s^{\alpha_1-1}x(0)+L[\beta x_n(\varepsilon)y_n(\varepsilon)]\right).$$

For $\lambda(s)=-\frac{1}{s^{\alpha_1}}$ and taking inverse Laplace,

$$x_{n+1}(\varepsilon)=L^{-1}\left(s^{-1}x(0)-\frac{1}{s^{\alpha_1}}L[\beta x_n(\varepsilon)y_n(\varepsilon)]\right).$$

Equation (16)'s correction function can be written as

$$Y_{n+1}(s)=Y_n(s)+\lambda(s)\left(s^{\alpha_2}Y_n(s)-s^{\alpha_2-1}y(0)-L[\beta x_n(\varepsilon)y_n(\varepsilon)-ky_n(\varepsilon)]\right).$$

For $\lambda(s)=-\frac{1}{s^{\alpha_2}}$, and taking inverse Laplace to equation, we get

$$y_{n+1}(\varepsilon)=L^{-1}\left(s^{-1}y(0)+\frac{1}{s^{\alpha_2}}L[\beta x_n(\varepsilon)y_n(\varepsilon)-ky_n(\varepsilon)]\right).$$

Further

$$Z_{n+1}(s)=Z_n(s)+\lambda(s)\left(s^{\alpha_3}Z_n(s)-s^{\alpha_3-1}z(0)-L[ky_n(\varepsilon)]\right).$$

For $\lambda(s)=-\frac{1}{s^{\alpha_3}}$, and taking inverse Laplace to equation, we get

$$z_{n+1}(\varepsilon)=L^{-1}\left(s^{-1}z(0)+\frac{1}{s^{\alpha_3}}L[ky_n(\varepsilon)]\right).$$

Case I: when n=0, equations give first iteration

$$x_1(\varepsilon)=L^{-1}\left(s^{-1}\delta-\frac{1}{s^{\alpha_1}}L[\beta\delta\gamma]\right),$$

$$x_1(\varepsilon)=\delta-(\beta\delta\gamma)\frac{\varepsilon^{\alpha_1}}{\Gamma(\alpha_1+1)}.$$

$$y_1(\varepsilon)=L^{-1}\left(s^{-1}\gamma+\frac{1}{s^{\alpha_2}}L[\beta\delta\gamma-k\gamma]\right),$$

$$y_1(\varepsilon)=\gamma+[\beta\delta\gamma-k\gamma]\frac{\varepsilon^{\alpha_2}}{\Gamma(\alpha_2+1)}.$$

$$z_1(\varepsilon)=L^{-1}\left(s^{-1}\mu+\frac{1}{s^{\alpha_3}}L[k\gamma]\right),$$

$$z_1(\varepsilon)=\mu+(k\gamma)\frac{\varepsilon^{\alpha_3}}{\Gamma(\alpha_3+1)}.$$

Case II: when n=1, we can obtain next components for *x*(ε), *y*(ε), *z*(ε) as

$$x_2(\varepsilon)=L^{-1}\left(s^{-1}\delta-\frac{1}{s^{\alpha_1}}L[\beta x_1(\varepsilon)y_1(\varepsilon)]\right),$$

$$x_2(\varepsilon)=L^{-1}\left(s^{-1}\delta-\frac{\beta}{s^{\alpha_1}}L\left[\left(\delta-(\beta\delta\gamma)\frac{\varepsilon^{\alpha_1}}{\Gamma(\alpha_1+1)}\right)\left(\gamma+[\beta\delta\gamma-k\gamma]\frac{\varepsilon^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\right]\right),$$

$$x_2(\varepsilon)=\delta-\beta\delta\gamma\frac{\varepsilon^{\alpha_1}}{\Gamma(\alpha_1+1)}-\beta\delta[\beta\delta\gamma-k\gamma]\frac{\varepsilon^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}+\beta^2\delta\gamma^2\frac{\varepsilon^{2\alpha_1}}{\Gamma(2\alpha_1+1)}+\beta^2\delta\gamma[\beta\delta\gamma-k\gamma]$$

$$\frac{\Gamma(\alpha_1+\alpha_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(2\alpha_1+\alpha_2+1)}\varepsilon^{2\alpha_1+\alpha_2}. \qquad 18$$

Similarly

$$y_2(\varepsilon)=\gamma+\beta\delta\gamma\frac{\varepsilon^{\alpha_2}}{\Gamma(\alpha_2+1)}+\beta\delta[\beta\delta\gamma-k\gamma]\frac{\varepsilon^{2\alpha_2}}{\Gamma(2\alpha_2+1)}-\beta^2\gamma^2\delta\frac{\varepsilon^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1+\alpha_2+1)}-\beta^2\delta\gamma[\beta\delta\gamma-k\gamma]$$

$$\frac{\Gamma(\alpha_1+\alpha_2+1)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)\Gamma(2\alpha_2+\alpha_1+1)}\varepsilon^{2\alpha_2+\alpha_1}-k\gamma\frac{\varepsilon^{\alpha_2}}{\Gamma(\alpha_2+1)}-k[\beta\delta\gamma-k\gamma]\frac{\varepsilon^{2\alpha_2}}{\Gamma(2\alpha_2+1)}. \qquad 19$$

And

$$z_2(\varepsilon) = \mu + k\gamma \frac{\varepsilon^{\alpha_3}}{\Gamma(\alpha_3+1)} + k[\beta\delta\gamma - k\gamma]\frac{\varepsilon^{\alpha_3+\alpha_2}}{\Gamma(\alpha_3+\alpha_2+1)}. \quad 20$$

In similar manner, rest of components can be obtained and $x_n(\varepsilon),\ y_n(\varepsilon),\ z_n(\varepsilon),$ rapidly converges to exact solution as $n \to \infty$, rapidly converges it means only few terms are required to get estimated solutions.

4. ALTERNATIVE EFFICIENT METHOD-1(AEM-1)

Consider nonlinear fractional derivative equation of the following type

$$D_0^{\alpha} y(\varepsilon) - y^2(\varepsilon) - 1 = 0, \quad 21$$

subjected to the conditions $y^{(i)}(0) = 0,\ (i = 0, 1, \ldots, m-1,),\ (m \geq \alpha > m-1,\ \varepsilon \geq 0).$

Taking αk^{th} order differentiation of (21)

$$D_0^{\alpha(k+1)} y(\varepsilon) - D_0^{\alpha k}\left(y^2(\varepsilon)\right) - D_0^{\alpha k}(1) = 0, \quad 22$$

where k = 0, 1, 2,....

Let us suppose

$$y(\varepsilon) = \sum_{n=1}^{\infty} a_n \varepsilon^{n\alpha}. \quad 23$$

Putting (23) in (22), we get

$$D^{\alpha(k+1)}\left(\sum_{n=1}^{\infty} a_n \varepsilon^{n\alpha}\right) - D^{\alpha k}\left(\sum_{n=1}^{\infty} a_n \varepsilon^{n\alpha}\right)^2 - D^{\alpha k}(1) = 0,$$

$$\sum_{n\geq 1} a_n \frac{\Gamma n\alpha + 1}{\Gamma((n-k-1)\alpha+1)}\varepsilon^{(n-k-1)\alpha} - \sum_{i\geq 1}\sum_{j\geq 1} a_i a_j \frac{\Gamma((i+j)\alpha+1)}{\Gamma((i+j-k)\alpha+1)}\varepsilon^{(i+j-k)\alpha} = \beta_k,$$

where $\beta_k = \begin{Bmatrix} 1 & k=0 \\ 0 & k \geq 1 \end{Bmatrix}.$

By comparing constant term (put $k = 0$), we get

$$a_1 \Gamma(\alpha+1) = 1 \Rightarrow a_1 = \frac{1}{\Gamma(\alpha+1)}.$$

By comparing coefficient of ε^{α}, (put $k = 0$), we get

$$a_2 \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)} = 0 \Rightarrow a_2 = 0.$$

By comparing coefficient of $\varepsilon^{p\alpha}$, (put $k = 0$), we get

$$a_{p+1} = \frac{\Gamma(p\alpha+1)}{\Gamma((p+1)\alpha+1)} \cdot \sum_{1\leq i<p} a_i a_{p-i}.$$

put p = 2,

$$a_3 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} a_1 a_1 \Rightarrow a_3 = \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} \frac{1}{(\Gamma(\alpha+1))^2}.$$

Putting different coefficients, we get

$$y(\varepsilon) = \frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)} + \frac{\varepsilon^{3\alpha}}{(\Gamma(\alpha+1))^2}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} + 2\frac{\Gamma(2\alpha+1)\Gamma(4\alpha+1)\,\varepsilon^{5\alpha}}{(\Gamma(\alpha+1))^3\Gamma(3\alpha+1)\Gamma(5\alpha+1)} + \ldots \quad 24$$

5. ALTERNATIVE EFFICIENT METHOD-2 (AEM-2)

AEM-2 is useful for solving following form of nonlinear equation

$$y = f + L(y) + N(y), \quad 25$$

where f is a recognized function, $L(y)$ and $N(y)$ linear and non- linear operators.

Since Caputo fractional derivation of order α is describe as $D^{\alpha} f(\varepsilon) = I^{m-\alpha}\frac{d^m f(\varepsilon)}{d\varepsilon^m}$, where $m \geq \alpha > m-1,$

further $I^{\alpha}(\varepsilon-b)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}(\varepsilon-b)^{\gamma+\alpha},$

$$(I^{\alpha} D^{\alpha} f)(\varepsilon) = f(\varepsilon) - \sum_{k=0}^{m-1} f^k(0)\frac{\varepsilon^k}{k!},$$

where $\gamma > -1,\ m-1 < \alpha \leq m.$

According to AEM-1, the series solution $y = \sum_{n=0}^{\infty} y_n,$

whose terms are determined by

$$y_0 = f,\ y_1 = L(y_0) + N(y_0),$$

$$y_2 = L(y_1) + N(y_0 + y_1) - N(y_0),$$

$$y_3 = L(y_2) + N(y_0 + y_1 + y_2) - N(y_0 + y_1),$$

$$y_{n+1} = L(y_n) + N(\sum_{i=0}^{n} y_n) - N(\sum_{i=0}^{n-1} y_n).$$

Let we solve following non -linear fractional equation by AEM 2

$$D^{\alpha} y(\varepsilon) = 1 + y^2(\varepsilon), \qquad (\varepsilon > 0, \qquad 0 < \alpha \le 1), \tag{26}$$

where $y(0) = 0$.

Equivalent integral equation of (1)

$$y(\varepsilon) = y(0) + I^{\alpha}(1) + I^{\alpha}(y^2(\varepsilon)),$$

$$y_0 = 0 + I^{\alpha}(1) = \frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)},$$

$$y_1 = I^{\alpha}(y_0^2) = I^{\alpha}\left(\frac{\varepsilon^{2\alpha}}{(\Gamma(\alpha+1))^2}\right) = \frac{\Gamma(2\alpha+1)\varepsilon^{3\alpha}}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2},$$

$$y_2 = I^{\alpha}(y_0 + y_1)^2 - I^{\alpha}(y_0^2)$$

$$= I^{\alpha}\left(\frac{\varepsilon^{\alpha}}{\Gamma(\alpha+1)} + \frac{\Gamma(2\alpha+1)\varepsilon^{3\alpha}}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2}\right)^2 - I^{\alpha}\left(\frac{\varepsilon^{2\alpha}}{(\Gamma(\alpha+1))^2}\right)$$

$$= I^{\alpha}\left[\frac{\varepsilon^{2\alpha}}{(\Gamma(\alpha+1))^2}\right] + I^{\alpha}\left[\frac{(\Gamma 2\alpha+1)^2 \quad \varepsilon^{6\alpha}}{(\Gamma 3\alpha+1)^2 \ (\Gamma(\alpha+1))^4}\right] + 2I^{\alpha}\left[\frac{\Gamma(2\alpha+1) \quad \varepsilon^{4\alpha}}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^3}\right] - I^{\alpha}\frac{\varepsilon^{2\alpha}}{((\Gamma\alpha+1))^2},$$

$$y_2(\varepsilon) = \frac{(\Gamma(2\alpha+1))^2 \ \Gamma(6\alpha+1)}{(\Gamma(3\alpha+1))^2 (\Gamma(\alpha+1))^4} \frac{\varepsilon^{7\alpha}}{\Gamma(7\alpha+1)} + 2\frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^3} \frac{\Gamma(4\alpha+1) \ \varepsilon^{5\alpha}}{\Gamma(5\alpha+1)}.$$

$$y(\varepsilon) = y_0(\varepsilon) + y_1(\varepsilon) + y_2(\varepsilon) + \ldots,$$

$$y(\varepsilon) = \frac{\varepsilon^{\alpha}}{\Gamma(1+\alpha)}$$

$$+ \frac{\Gamma(2\alpha+1) \quad \varepsilon^{3\alpha}}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^2} + \frac{(\Gamma(1+2\alpha))^2 \ \Gamma(6\alpha+1)}{(\Gamma(3\alpha+1))^2(\Gamma(\alpha+1))^4} \frac{\varepsilon^{7\alpha}}{\Gamma(7\alpha+1)} + 2\frac{\Gamma(1+2\alpha)}{\Gamma(3\alpha+1)(\Gamma(\alpha+1))^3} \frac{\Gamma(4\alpha+1) \ \varepsilon^{5\alpha}}{\Gamma(5\alpha+1)} \ldots \tag{27}$$

which is in best agreement with exact result merely at second approximate solution. Hence this new method is comparative effective and fast converging method.

6. CONCLUSION

In order to show, appraise, and enhance the research objectives outlined above, the most appropriate and powerful arithmetical techniques for nonlinear fractional ordinary differential equations are being developed. Several effective numerical and analytical strategies for fractional derivative equations, both linear and non-linear, are presented and applied in mathematical models. Fractional derivative problems may be solved using the MVIM and power series techniques. In the second iteration, the proposed approach is able to provide results in a series form that converges quickly, and the precise results are acquired. There is a long-term relationship between fractional time derivatives and solution in entire instances, according to the results. The effectiveness of two new efficient approaches is shown by implementing them on a class of nonlinear fractional differential equations. Solving the composite fractional relaxation equation using the fractional power series approach is the focus of this section. In order to solve the composite fractional oscillation equation and the SIR model, the Modified variation iteration method (MVIM) is used. Solution behavior for fractional and classical order 1 is illustrated in graphical form. To solve nonlinear fractional equations and fractional order gas dynamics equations, several innovative efficient numerical techniques AEM-1 and AEM-2 are examined.

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