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New Convergence Techniques for Identifying with Fixed Point by Nonexapnsive Mappings

D. P. Shukla[1]* Vivek Tiwari[2]

[1] Department of Mathematics/Computer Science, Govt. Science College, Rewa (M.P.) 486001, India

[2] Research Scholar, Govt. Science College, Rewa (M.P.) 486001, India

Abstratct – This paper aimed to be generating new iterative outcomes and double step iterative techniques for recognizing of fixed points of nonexpansive mappings in Banach space. Encourage we demonstrate another iterative procedure, which is finer than different other existing iterative methods.

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1. INTRODUCTION

We take a set E which is uniformly convex Banach space and E is the super set of C and C is closed convex set. In this paper, N indicates the set of all positive integers and G (T) = {x: Gx = x}. A mapping T: C $\rightarrow$ C is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\| \forall x, y \in$ N. For any arbitrary we can take $x_1 \in C$, Generate a sequence $\{x_n\}$, where x_n is characterized by positive integer n $\geq$ 1 as:

$$x_{n+1} = Tx_n, \qquad \text{(i)}$$

Hear (i) known as Picard sequence.

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \qquad \text{(ii)}$$

Here (ii) known as Mann [8] sequence.

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \\ y_n = (1-\beta_n)x_n + \beta_n Tx_n \end{cases} \qquad \text{(iii)}$$

Here (iii) known as Ishikawa [5] sequence.

Ishikawa, Mann and other iteration methods have considered by few researchers for approximation fixed point of nonexpansive mapping [6, 11, 13-15].

Noor [9] defined iterative method by $x_1 \in C$, as follows:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Ty_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n Tz_n$$

$$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \forall n \geq 1, \qquad \text{(iv)}$$

Where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequence in (0, 1).

Agrawal [2] in 2007 constructs accompanying strategy:

$$x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n,$$

$$y_n = (1-\beta_n)x_n + \beta_n Tx_n, \qquad \text{(v)}$$

Where $\{\alpha_n\}$ and $\{\beta_n\}$ are in (0, 1). They demonstrated that this scheme converges at equivalence comparatively as Picard schemes and it's comparatively finer than Mann iterative schemes for contration mappings.

Abbas et. al. [1] constructs the other following schemes, where sequence $\{x_n\}$ is generated from any arbitrary $x_1 \in C$

$$x_{n+1} = (1-\alpha_n)Ty_n + \alpha_n Tz_n,$$

$$y_n = (1-\beta_n)Tx_n + \beta_n Tz_n$$

$$z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \qquad \text{(vi)}$$

Where $\{\alpha_n\},\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in (0, 1). They proved that this scheme converges as finer than iterative techniques (v) [2].

Now, motivated by above all techniques we build another iterative process for determining the fixed point of nonexpansive mapping. Where sequence $\{x_n\}$ is generated by $x_1 \in C$ and written as follows:

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n, \end{aligned}\right\} \qquad \text{(vii)}$$

The primary goal of this article to satisfies convergent outcomes for nonexpansive mappings by iteration (vii). We also show that iteration (vii) converges faster than various other iterative schemes.

2. PRELIMINARIES:

Consider K is a Banach space and S_K = $\{x \in K : \|x\| = 1\}$ is sphere on K having magnitude is identity. For all $\delta \in (0,1)$ and $x, y \in S_K$ with $x \neq y$, if $\|(1-\delta)x + \delta y\| < 1$, then is said to be convex i.e. strictly convex Banach space. $\|x\| = \|y\| = \|\alpha x + (1-\alpha)y\|$ and $\alpha \in$ (0,1), then x= y.

Then space K is called smooth if $\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$.

The space K is satisfies the Opial's condition [10] for every sequence $\{x_n\}$ in K, i.e

$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ $\forall y \in K$ with $x \neq y$.

Definition 2.1 Assume that $\{r_n\}$ and $\{s_n\}$ are sequences that both sequences have limit point i.e both are converges to r and s. if $\lim_{n\to\infty} \frac{|r_n - r|}{|s_n - s|} = 0$, then $\{r_n\}$ converges faster than $\{s_n\}$.

Definition 2.2 Consider two fixed point iteration processes $\{p_n\}$ and $\{q_n\}$, both are conversing to common fixed point at t, then the error estimates are as below:

$$\|p_n - t\| \le r_n$$

$$\|q_n - t\| \le s_n \text{ both are defined for } n \ge 1,$$

Here $\{r_n\}$ and $\{s_n\}$ are two real sequences of positive number tends to 0. If $\{r_n\}$ converges finer than $\{s_n\}$, then $\{p_n\}$ is also finer than $\{u_n\}$ to t.

Here we will characterize a few lemmas for further uses in this paper which are as follows:

Lemma2.3 [4] Assume C is a nonempty closed convex subset of a uniformly convex Banach space K and T is a nonexpansive mapping on C. So I – T is demeclosed at 0.

Lemma 2.4[12] Consider E is a uniformly comvex Banach space and $0 < p \le t_n \le q < 1$. Now $\{x_n\}$ and $\{y_n\}$ is the two sequences of E such that $\limsup_{n\leftarrow\infty} \|x_n\| \le r, \limsup_{n\to\infty} \|y_n\| \le r$ and

$\limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r,$ hold for some $r \ge 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.5 [2] Assume that E be a reflexive Banach space fulfilling the Opial condition and a function T: C $\to$X such that I- T demiclosed at 0 and F (T) $\neq \phi$ where C is a convex subset of E. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - p\|$ exist for all $p \in F(T)$. Then $\{x_n\}$ converges to a fixed point of T.

3. CONVERGENCE RATE

In this part, we will prove that iteration method (vii) converges finer than the other methods.

Theorem 3.1 Let we define a set E such that set E is normed linear space and there is the subset C of E, which is nonvoid closed convex set and let T be a constructive mapping included a factor $k \in (0,1)$ and fixed point p. Let $\{u_n\}$ be characterized by the iteration techniques (vi) and $\{x_n\}$ by (vii), where $\{\alpha_n\}$ and $\{\beta_n\}$ are in

$[\varepsilon, 1-\varepsilon]\ \forall n \in N$ and for some ε in (0, 1). So $\{x_n\}$ converges finer from $\{u_n\}$.

Proof: As demonstrated in Theorem 3 of M. Abbas and T. Nazir[1].

$$\|u_{n+1} - p\| \le k^n[1-(1-k)\alpha\beta\gamma]^n \|u_1 - p\|,\ n \in N.$$

Let $a_n = k^n\{1-(1-k)\alpha\beta\gamma\}^n \|u_1 - p\|$

$$\|y_n - p\| = \|(1-\beta_n)x_n + \beta_n Tx_n - p\|$$

$$\le (1-\beta_n)\|x_n - p\| + \beta_n \|Tx_n - p\|$$

$$= (1-(1-k)\beta_n)\|x_n - p\|$$

Thus

$$\|x_{n+1} - p\| = \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\|$$

$$\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n \|Ty_n - p\|$$

$$\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n \|(1-\beta_n)x_n + \beta_n Tx_n - p\|$$

$$\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n(1-\beta_n)\|x_n - p\| + \alpha_n\beta_n\|Tx_n - p\|$$

$$\le k\{1-\alpha_n + \alpha_n(1-(1-k)\beta_n)\}\|x_n - p\|$$

$$= k\{1-(1-k)\alpha_n\beta_n\}\|x_n - p\|.$$

Suppose $b_n = k^n\{1-(1-k)\alpha\beta\}^n \|x_1 - p\|$

$$\frac{b_n}{a_n} = \frac{k^n\{1-(1-k)\alpha\beta\}^n\|x_1 - p\|}{k^n\{1-(1-k)\alpha\beta\gamma\}^n\|u_1 - p\|} \to 0$$

as $n \to \infty$.

Consequently $\{x_n\}$ converges finer than $\{u_n\}$.

Theorem 3.2 Here we consider a self mapping T on set C and set C is nonvoid closed set. Now assume E as a normed linear space E where E is the super set of C, a sequence $\{x_n\}$ defined by (vii) and F (T) $\neq \phi$. Then $\lim_{n\to\infty}\|x_n - p\|$ exist for each $p \in F(T)$.

Proof. Now by (vii), we have

$$\|y_n - p\| = \|(1-\beta_n)x_n + \beta_n Tx_n - p\|$$

$$\le (1-\beta_n)\|x_n - p\| + \beta_n\|Tx_n - p\|$$

$$\le (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\|$$

$$= \|x_n - p\| \qquad \text{(viii)}$$

so

$$\|x_{n+1} - p\| = \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\|$$

$$\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\|$$

$$\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\|$$

$$= \|x_n - p.\| \qquad \text{(ix)}$$

Thus $\lim_{n\to\infty}\|x_n - p\|$ exists $\forall p \in F(T)$.

Theorem 3.3 Let we have a set E which is uniformly Banach space and E is the super set of C i.e $C \subseteq E$ where C is nonempty closed convex set. Let T be a nonexpansive self-mapping on C, and a sequence $\{x_n\}$ provided by (vii) and F (T) $\neq \phi$ then $\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$

Proof By theorem 3.2 we have

$\lim_{n\to\infty}\|x_n - p\|$ is exists.

Then assume $\lim_{n\to\infty}\|x_n - p\| = c.$

By from (viii) and (ix) we have

$\limsup_{n\to\infty}\|x_n - p\| \le c$, and

$$\limsup_{n\to\infty}\|y_n - p\| \le c \qquad \text{(x)}$$

But T is a nonexpansive map so we have

$\|Tx_n - p\| \le \|x_n - p\|$ and $\|Ty_n - p\| \le \|y_n - p\|$

After getting limsup on both sides,

$$\limsup_{n\to\infty}\|Tx_n - p\| \le c \quad \text{(xi)}$$

And

$$\limsup_{n\to\infty}\|Ty_n - p\| \le c \quad \text{(xii)}$$

Since

$$\lim_{n\to\infty}\|x_{n+1} - p\| = \lim_{n\to\infty}\|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\|$$

Hence by the using of lemma 2.4

$$\lim_{n\to\infty}\|x_n - y_n\| = 0.$$

Now,

$$\|x_{n+1} - p\| = \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\|$$

$$\le \|Tx_n - p\| + \alpha_n\|Tx_n - Ty_n\|$$

Yield that

$$\limsup_{n\to\infty}\|Tx_n - p\| \ge c \quad \text{(xiii)}$$

Now form (xi) and (xiii)

$$\lim_{n\to\infty}\|Tx_n - p\| = c,$$

additionally we have

$$\|Ty_n - p\| \le \|Ty_n - Tx_n\| + \|Tx_n - p\|$$

$$\le \|Ty_n - Tx_n\| + \|y_n - p\|$$

and

$$\limsup_{n\to\infty}\|Ty_n - p\| \ge c \quad \text{(xiv)}$$

So from (xii) & (xiv) and

$$\liminf_{n\to\infty}\|y_n - p\| = c.$$

Now by utilizing the lemma 2.4, from we have

$$\lim_{n\to\infty}\|y_n - Ty_n\| = 0.$$

since $c = \lim_{n\to\infty}\|y_n - p\|$

$$= \lim_{n\to\infty}\|(1-\beta_n)x_n + \beta_n Tx_n - p\|$$

$$= \lim_{n\to\infty}\|(1-\beta_n)(x_n - p) + \beta_n Tx_n - p\|$$

Hence by the lemma 2.4

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0.$$

Hence the theorem is verified.

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Corresponding Author

D. P. Shukla*

Department of Mathematics/Computer Science, Govt. Science College, Rewa (M.P.) 486001, India

E-Mail – shukladpmp@gmail.com