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An Analysis upon Various Aspects of Metric Spaces in Real Analysis: Some Solutions

Sandeep*

Assistant Professor in Mathematics, C. R. S. U., Jind, Haryana

Abstractt – This paper will introduce the reader to the concept of metrics and metric spaces. A lot emphasis has been given to motivate the ideas under discussion to help the reader develop skill in using his imagination to visualize the abstract nature of the subject. Variety of examples along with real life applications have been provided to understand and appreciate the beauty of metric spaces.

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INTRODUCTION

Many of the arguments you have seen in several variable calculus are almost identical to the corresponding arguments in single variable calculus, especially arguments concerning convergence and continuity. The reason is that the notions of convergence and continuity can be formulated in terms of distance, and that the notion of distance between numbers that you need in single variable theory, is very similar to the notion of distance between points or vectors that you need in the theory of functions of severable variables. In more advanced mathematics, we need to find the distance between more complicated objects than numbers and vectors, e.g. between sequences, sets and functions. These new notions of distance leads to new notions of convergence and continuity, and these again lead to new arguments surprisingly similar to those you have already seen in single and several variable calculus.

After a while it becomes quite boring to perform almost the same arguments over and over again in new settings, and one begins to wonder if there is general theory that covers all these examples { is it possible to develop a general theory of distance where we can prove the results we need once and for all? The answer is yes, and the theory is called the theory of metric spaces.

A metric space is just a set X equipped with a function d of two variables which measures the distance between points: d(x; y) is the distance between two points x and y in X. It turns out that if we put mild and natural conditions on the function d, we can develop a general notion of distance that covers distances between numbers, vectors, sequences, functions, sets and much more. Within this theory we can formulate and prove results about convergence and continuity once and for all. The purpose of this chapter is to develop the basic theory of metric spaces. In later chapters we shall meet some of the applications of the theory. We now introduce the idea of a metric space, and show how this concept allows us to generalise the notion of continuity. We will then concentrate on looking at some examples of metric spaces.

DEFINITIONS AND EXAMPLES

As already mentioned, a metric space is just a set *X* equipped with a function $d: X \times X \to \mathbb{R}$ that measures the distance *d(x, y)* between points $x, y \in X$. For the theory to work, wc need the function *d* to have properties similar to the distance functions we are familiar with. So what properties do wc expect from a measure of distance?

First of all, the distance *d(x, y)* should be a nonnegative number, and it should only be equal to zero if *x* = *y*. Second, the distance *d(x,y)* from *x* to *y* should equal the distance *d(y, x)* from *y* to *x.* Note that this is not always a reasonable assumption - if *we,* e.g., measure the distance from *x* to *y* by the time it takes to walk from *x* to *y, d(x,y)* and *d(y,x)* may be different - but we shall restrict ourselves to situations where the condition is satisfied. The third condition we shall need, says that the distance obtained by going directly from *x* to *y,* should always be less than or equal to the distance we get when we go via a third point *z,* i.e.

$$d(x,y) \le d(x,z) + d(z,x)$$

It turns out that these conditions are the only ones we need, and we sum them up in a formal definition.

Definition 1. *A* metric space (*X,d) consists of a non-empty set X and a function* $d: X \times X \to [0, \infty)$ *such that:*

(i) *(Positivity) For all* $x, y \in X,$ *we have* $d(x, y) \geq 0$ *with equality if and only if x = y.*

(ii) *(Symmetry) For all* $x, y \in X,$ *we have* $d(x, y) = d(y, x).$

(iii) *(Triangle inequality) For all* $x, y, z \in X,$ *we have* $d(x, y) \leq d(x, z) + d(z, y)$

A function d satisfying conditions (i)-(iii) is called a metric *on X.*

Comment: When it is clear - or irrelevant - which metric *d we* have in mind, we shall often refer to "the metric space *X"* rather than "the metric space *(X.d*)".

Let us take a look at some examples of metric spaces.

Example 1: If we let $d(x, y) = |x - y|,$ then $(\mathbb{R}, d)$ is a metric space. The first two conditions are obviously satisfied, and the third follows from the ordinary triangle inequality for real numbers:

$$d(x, y) = |x - y| = |(x - z) + (z - y)$$

$$| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$$

Example 2: If we let

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

then $(\mathbb{R}^n, d)$ is a metric space. The first two conditions are obviously satisfied, and the third follows from the triangle inequality for vectors the same way as above :

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \|(\mathbf{x} - \mathbf{z}) + (\mathbf{z} - \mathbf{y})$$

$$\| \leq \|\mathbf{x} - \mathbf{z}\| + \|\mathbf{z} - \mathbf{y}\| = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$$

Example 3: Assume that we want to move from one point $\mathbf{x} = (x_1, x_2)$ in the plane to another $\mathbf{y} = (y_1, y_2),$ but that we are only allowed to move horizontally and vertically. If we first move horizontally from (x_1, x_2) to (y_1, x_2) and then vertically from (y_1, x_2) to (y_1, y_2), the total distance is $d(\mathbf{x}, \mathbf{y}) = |y_1 - x_1| + |y_2 - x_2|$

This gives us a metric on R^2 which is different from the usual metric in Example '2. It is often referred to as the *Manhattan metric* or the *taxi cab metric.*

Also in this case the first two conditions of a metric space are obviously satisfied. To prove the triangle inequality, observe that for any third point $\mathbf{z} = (z_1, z_2),$

we have

$$\begin{aligned} d(\mathbf{x}, \mathbf{y}) &= |y_1 - x_1| + |y_2 - x_1| = \\ &= |(y_1 - z_1) + (z_1 - x_1)| + |(y_2 - z_2) + (z_2 - x_2)| \leq \\ &\leq |y_1 - z_1| + |z_1 - x_1| + |y_2 - z_2| + |z_2 - x_2| = \\ &= |z_1 - x_1| + |z_2 - x_2| + |y_1 - z_1| + |y_2 - z_2| = \\ &= d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}) \end{aligned}$$

where we have used the ordinary triangle inequality for real numbers to get from the second to the third line.

GEOMETRY OF METRIC SPACES

Before we look at what it means for a sequence to be convergent with respect to a given metric, we spend a little time discussing one way of gaining some understanding about the geometric meaning of a given metric.

In the last subsection, we met three different metrics: the discrete metric, the taxicab metric on the plane and a mixed metric on the plane (which was formed from the usual distance in *R* together with the discrete metric).

An easy way to gain some insight into the behaviour of a metric is to look at the *balls* around a given point. For the usual Euclidean distance in $\mathbb{R}^n,$ a ball of radius *r* around a point $a \in \mathbb{R}^n$ consists of all those points whose distance from a is at most r, and this definition naturally extends to general metric spaces. However, in the following definition we take care to distinguish between balls that include points at exactly distance *r* from the centre a and those that do not.

Definition 1 ***Open and closed balls***

Let (*X*, *d)* be a metric space, and let $a \in X$ and $r \geq 0.$

The open ball of radius *r* with centre *a* is the set

$$B_d(a, r) = \{x \in X : d(a, x) < r\}.$$

The closed ball of radius *r* with centre *a* is the set

$$B_d[a, r] = \{x \in X : d(a, x) \leq r\}.$$

The sphere of radius r with centre *a* is the set

$$S_d(a,r) = \{x \in X : d(a,x) = r\}.$$

When *r* = 1, these sets are called respectively the unit open ball with centre *a,* the unit closed ball with centre *a* and the unit sphere with centre *a.*

Worked Exercise 1

Let *{X.d)* be a metric space, and let $a \in X$. Show that $B_d(a,0) = \emptyset,\quad B_d[a,0] = \{a\}$ and $S_d(a,0) = \{a\}.$

Solution

It follows from (Ml) that

$$B_d(a,0) = \{x \in X : d(a,x) < 0\} = \emptyset,$$

$$B_d[a,0] = \{x \in X : d(a,x) \le 0\} = \{x \in X : d(a,x) = 0\} = \{a\},$$

and

$$S_d(a,0) = \{x \in X : d(a,x) = 0\} = \{a\}.$$

We now discover what open balls, closed balls and spheres look like for some of the metric spaces we have met already.

Let us start by determining the open and closed balls for the discrete metric, d_0

Worked Exercise 2

Let *X* be a non-empty set and $a \in X$. Determine $B_{d_0}(a,r)$ for $r \ge 0$ **Solution**

Let $a \in X$ and suppose that $r > 0.$

Since $B_{d_0}(a,r) = \{x \in X : d_0(a,x) < r\}$ and $d_0(a,x) = 1$ unless a=x

(when it is 0), we conclude that

$$B_{d_0}(a,r) = \begin{cases} \emptyset, & \text{if } r = 0, \\ \{a\}, & \text{if } 0 < r \le 1, \\ X, & \text{if } r > 1. \end{cases}$$

Worked Exercise 3

Consider the metric space $(\mathbb{R}^2, e_1)$- that is, the plane with the taxicab metric. Find the unit open ball $B_{e_1}(\mathbf{0},1)$

Solution

The centre is $\mathbf{0} = (0,0),$ and we want to find all points $\mathbf{x} = (x_1, x_2)$ that satisfy

$$e_1(\mathbf{0},\mathbf{x}) = |x_1| + |x_2| < 1.$$

We first consider points in the first quadrant, where $x_1, x_2 \ge 0.$

We want to find those points where $x_1 + x_2 < 1.$ Consider the line $x_1 + x_2 = 1,$ or equivalently $x_2 = 1 - x_1.$ In the first quadrant, this line connects the points (0,1) and (1,0). The points on this line segment have coordinates $(x_1, 1 - x_1).$ All points below the line segment have coordinates (x_1, x_2) with $x_2 < 1 - x_1$ and all points on or above it have coordinates (x_1, x_2) with $x_2 \ge 1 - x_1.$ Hence the points where $x_1 + x_2 < 1$ are those strictly below the line segment, making up the shaded region.

By use of a similar argument for each of the other three quadrants, or by appealing to the symmetry of the situation, we obtain triangular regions in each quadrant. Combining these, we obtain the diamond-shaped region; the open ball $B_{e_1}(\mathbf{0},1)$ is the set of points strictly inside this diamond, shown shaded in the figure. The dashed boundary indicates that it is not included in the set.

SEQUENCES IN METRIC SPACES

Now that we have several examples of metric spaces available to us, we return to the problem of defining continuous functions between metric spaces.

Since the definition of a general metric space is modelled on the properties of the Euclidean metric $d^{(n)}$ on $\mathbb{R}^n,$ and we defined continuity of functions between Euclidean spaces in terms of convergent sequences, it is natural to attempt to extend our ideas about convergent sequences in $\mathbb{R}^n$ to general metric spaces. In fact, we did much of the hard work when we generalised from the notion of convergence for real-valued sequences to that of convergence of sequences in $\mathbb{R}^n;$ it is now only a short step to develop these concepts for the metric space setting.

We observed that a real sequence can be thought of as a function $a : \mathbb{N} \to \mathbb{R},$ given by $n \mapsto a_n.$ Note that the only role played by $\mathbb{R}$ here is as the codomain of the function $a : \mathbb{N} \to \mathbb{R};$ the structure of $\mathbb{R}$ becomes relevant only when convergence is considered. Since the codomain of a function is simply a set, the following definition is a natural generalisation.

Definition 1 ***Sequence in a metric space***

Let A be a set. A sequence in *X* is an unending ordered list of elements of *X:*

$a_1, a_2, a_3, \ldots.$

The element a* is the *kth* term of the sequence, and the whole sequence is denoted by (a_k), $(a_k)_{k=1}^{\infty}$ or $(a_k)_{k\in\mathbb{N}}$.

Note that this definition of a sequence does not require that we impose any additional structure (such as a metric) on the set *X.*

The definition of what it means for a sequence to converge in a metric space *(X, d)* is closely based on the definition of convergence in $\mathbb{R}^n$

Definition 2 ***Convergence in a metric space***

Let *(X.d)* be a metric space. A sequence *(a$_k$)* in *X d-*converges to $a \in X$ if the sequence of real numbers $(d(a_k, a))$ is a null sequence.

We write $a_k \xrightarrow{d} a$ as $k \to \infty$, or simply $a_k \to a$ if the context is clear.

We say that the sequence (a_k) is convergent in *(X,d)* with limit a.

A sequence that does not converge (with respect to the metric *d)* to any point in *X* is said to be d-divergent..

Exercise 1

Let $(\mathbb{R}^2, e_1)$ be the plane with the taxicab metric, and let (a_n) be the sequence given by Show that (a_k) converges to (1,2) with respect to *e$_1$.* $a_k = (1 + \frac{1}{2k}, 2 - \frac{1}{2k})$.

Convergent sequences in $(\mathbb{R}^n, d^{(n)})$ have unique limits - that is, a sequence cannot simultaneously converge to two different limits. The next result establishes this as a fact in any metric space.

Theorem 1 ***Uniqueness of limits in a metric space***

Let (*X*, *d)* be a metric space and let $a, b \in X$. If (a_k) is a sequence in *X* that d-converges to both *a* and *b,* then *a* = *b.*

Proof We use proof by contradiction.

Suppose that the sequence (a_k) d-converges to both *a* and *b* in *X*, with $a \neq b$. Thenby property (Ml) of Definition 1.1 for d, $d(a,b) > 0$ and so if we let $\varepsilon = \frac{1}{2}d(a,b)$, then $\varepsilon > 0$.

Since we are supposing that the sequence (a_k) converges to both *a* and *b*, the sequences of real numbers $(d(a_k, a))$ and $(d(a_k, b))$ are both null. Hence there is $N \in \mathbb{N}$ for which $d(a_k, a) < \varepsilon$ and $d(a_k, b) < \varepsilon$ whenever $k > N$.

The Triangle Inequality (property (M3) for d) tells us that, for each $k > N$,

$$\begin{aligned} d(a,b) &\leq d(a, a_k) + d(a_k, b) < \varepsilon + \varepsilon = 2\varepsilon \\ &= d(a,b), \end{aligned}$$

by the definition of ε. But this is impossible; hence our initial assumption that a sequence could converge to two distinct limits must be wrong. We conclude that any d-convergent sequence has a unique limit.

CONTINUITY IN METRIC SPACES

Now that we know what it means for a sequence to converge in a metric space, we can formulate a definition of continuity for functions between metric spaces.

Definition 1 ***Continuity for metric spaces***

Let (*X*, *d)* and (*Y*, e) be metric spaces and let $f: X \to Y$ be a function.

Then *f* is (d, e)-continuous at $a \in X$ if:

Whenever (a_k) is a sequence in *X* for which $a_k \xrightarrow{d} a$ as $k \to \infty$, then the sequence $f(a_k) \xrightarrow{e} f(a)$ as $k \to \infty$.

If *f* does not satisfy this condition at some $a \in X$- that is, there is a sequence (x_k) in *X* for which $x_k \to a$ as $k \to \infty$ but $f(x_k)$ does not converge to $f(a)$ then we say that *f* is (d, e)-discontinuous at a.

A function that is continuous at all points of *X* is said to be (d, e)-continuous on *X* (or simply continuous, if no ambiguity is possible).

Our next worked exercise shows that this definition can make some surprising functions continuous.

Worked Exercise 1

Let $f: \mathbb{R} \to \mathbb{R}$ be a function and let $a \in \mathbb{R}$. Prove that *f* is always $(d_0, d^{(1)})$-continuous at *a.*

Solution

Let $a \in \mathbb{R}$ and suppose that (x_k) is a sequence in $\mathbb{R}$ that is d_0-convergent to *a.*

Then we deduce that there is $N \in \mathbb{N}$ so that for $k > N$, $x_k = a$. But then for $k > N$, $f(x_k) = f(a)$ and so for such *k*, $d^{(1)}(f(x_k), f(a)) = |f(a) - f(a)| = 0.$ Hence $(d^{(1)}(f(x_k), f(a)))$ is a

real null sequence and we conclude that *f* is $(d_0, d^{(1)})$-continuous at *a.*

This is a rather artificial example and it tells us that every function from $\mathbb{R}$ to $\mathbb{R}$ is $(d_0, d^{(1)})$-continuous on $\mathbb{R}.$ However, it does illustrate that our intuitive notion of what continuity means breaks down when looking at metrics different from the Euclidean ones, and so highlights the importance of working from the definition.

Worked Exercise 2

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x_1, x_2) = (x_1, 2x_2).$ Prove that *f* is $(d^{(2)}, e_1)$-continuous on $\mathbb{R}^2.$

Solution

Let $a = (a_1, a_2) \in \mathbb{R}^2.$ We must show that if (x*) is a sequence in the plane that $d^{(2)}$-converges to a, then $(e_1(f(x_k), f(a)))$ is a real null sequence.

Suppose that $(x_k = (x_{1,k}x_{2,k}))$ is a sequence in the plane that $d^{(2)}$-converges to a, that is, a sequence for which $(d^{(2)}(x_k, a))$ is a real null sequence. Then

$e_1(f(x_k), f(a)) = |x_{1,k} - a_1| + |2x_{2,k} - 2a_2|,$ by definition of *f* and e_1

$$\begin{aligned} &= |x_{1,k} - a_1| + 2|x_{2,k} - a_2| \\ &\leq 2(|x_{1,k} - a_1| + |x_{2,k} - a_2|) \\ &\leq 2(d^{(2)}(x_k, a) + d^{(2)}(x_k, a)) = 4d^{(2)}(x_k, a). \end{aligned}$$

But $e_1(f(x_k), f(a)) \geq 0$ for every *k* and we are assuming that $(d^{(2)}(x_k, a))$ is a real null sequence. Hence by the Squeeze Rule, $e_1(f(x_k), f(a)) \to 0$ as $k \to \infty.$ That is, $(e_1(f(x_k), f(a)))$ is also a real null sequence and so *f* is $(d^{(2)}, e_1)$-continuous at *a.*

Since $a \in \mathbb{R}^2$ was an arbitrary point in $\mathbb{R}^2,$ we conclude that *f* is $(d^{(2)}, e_1)$-continuous on $\mathbb{R}^2.$

At the moment our stock of metric spaces is quite small: Euclidean spaces, the plane with the taxicab metric, the plane with a particular 'mixed[5] metric, and arbitrary sets with the discrete metric. In the next chapter we will look at more examples of metric spaces and examine further the notion of continuity. What we can do at this point, though, is prove a useful result that applies to all continuous functions and which is an extension of the Composition Rule for continuous functions between Euclidean spaces.

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Corresponding Author

Sandeep*

Assistant Professor in Mathematics, C. R. S. U., Jind, Haryana