Studies of Extension of Fixed Point Theorem of Rhoades Using Ishikawa Iteration Process

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Abstract – In this paper we have extension of fixed point theorem of Rhoades using ishikawa iteration process has been proved.

Keywords: Fixed point, ishikawa iteration process, L_P Space

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INTRODUCTION

If c be a non empty subset of x, where x be a Banach space. And let T a mapping from C to itself. The iteration scheme called Ishikawa Scheme is defined as follows:

$$X_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, n \geq 0 \qquad \ldots (I_1)$$

$$y_n = \beta_n T x_n + (1-\beta_n)x_n, n \geq 0 \qquad \ldots (I_2)$$

$$x \varepsilon C \qquad \ldots (I_3)$$

In above Ishikawa scheme, $\{\alpha_n\}, \{\beta_n\}$ satisfy

(i) $0 \leq \alpha_n \leq \beta_n \leq 1$ for all n, $\lim_{n\to\infty} \beta_n = 0$

And $\sum \alpha_n \beta_n = \infty$.

(ii) $\underline{\lim}\ \alpha_n = \alpha > 0$

(iii) $\overline{\lim}\ \beta_n = \beta < 1$.

There are following two contractive conditions to be used. There exists a constant K, $0 \leq k \leq 1$ such that for all x, y in x.

(A) $\|Tx - Ty\| \leq k \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\| + \|y-Tx\|\}$

(B) At least one of the following conditions holds:

(i) $\|x-Tx\| + \|y-Ty\| \leq a\|x-y\|, 1 \leq a \leq 2$, for each x, y in x,

(ii) $\|x-Tx\| + \|y-Ty\| \leq b[\|x-Ty\| + \|y-Tx\| + \|x-y\|] \frac{1}{2} \leq b \leq \frac{2}{3}$, and for each x, y in X.

(iii) $\|x-Tx\| + \|y-Ty\| + \|Tx-Ty\| < c[\|x-Ty\| + \|y-Tx\|] 1 \leq c \leq \frac{3}{2}$ and for each x, y in X.

(iv) $\|Tx-Ty\| \leq k \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|\}$, $[\|x-Ty\| + \|y-Tx\|]/2$, $0 \leq k < 1$ for each x, y in X.

In this paper it is shown that, for mapping T which satisfy conditions (A) or (B) above, if the sequence of Ishikawa iterates converges, it converges to the fixed point of T. These results extend the corresponding results of Rhoades [1] and Hicks and Kubicek [2].

Definition 1: A mapping T : $X \to X$ is called a quasicontraction if there exists a constant K, $0 \leq k < 1$ such that for each x, y ││ X, where X be a Banach space.

$$\|Tx - Ty\| \leq k \max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\|, \|y-Tx\|\}$$

Theorem 1: Suppose T: $C \to C$ be a mapping satisfying (A), $\{X_n\}$ the sequence of the Ishikawa scheme associated with T are such that {││$_n$} is bounded away from zero. If $\{x_n\}$ converges to p, then p is a fixed point of T, where X be a normed linear space and C be a closed convex subset of X.

Proof: We have from $\{I_1\}$ that

$$x_{n+1} - x_n = \alpha_n (Ty_n - x_n).$$

Since $x_n \to p, \| x_{n+1} - x_n \| \to 0.$ Since {| |$_n$} is bounded away from zero, $\| Ty_n - x_n \| \to 0$. It also follows that $\| p - Ty_n \| \to 0$. Since T satisfies (A) we have

$$\| Ty_n - Tx_n \| \le k \max \{ \| y_n - x_n \|, \| x_n - Tx_n \|, \| y_n - Ty_n \|, \| x_n - Ty_n \| + \| y_n - Tx_n \| \}$$

$$\| y_n - x_n \| = \| \beta_n Tx_n + (1-\beta_n)x_n - x_n \| \le \beta_n \| x_n - Tx_n \| \le \| x_n - Tx_n \| \le \| x_n - Ty_n \| + \| Ty_n - Tx_n \|;$$

$$\| y_n - Ty_n \| = \| \beta_n Tx_n + (1-\beta_n)x_n - Ty_n \| \le \beta_n \| Tx_n - Ty_n \| + (1-\beta_n) \| x_n - Ty_n \| \le \| x_n - Ty_n \| + \| Tx_n - Ty_n \|;$$

$$\| y_n - Tx_n \| = \| \beta_n Tx_n + (1-\beta_n)x_n - Tx_n \| \le \| x_n - Ty_n \| + \| Ty_n - Tx_n \|.$$

Thus

$$\| Tx_n - Ty_n \| \le \frac{2k}{1-k} \| x_n - Ty_n \|.$$

We have by taking the limit as $n \to \infty$,

$$\| Tx_n - Ty_n \| \to 0.$$

It follows that

$$\| x_n - Tx_n \| \le \| x_n - Ty_n \| + \| Ty_n - Tx_n \|$$

$$\text{and} \quad \| p - Tx_n \| \le (\| p - x_n \| + \| x_n - Tx_n \|) \to 0 \text{ and } n \to \infty$$

Using the definition (1) of T and the triangle inequality, we have

$$\| Tx_n - Tp \| \le k \max \{ \| x_n - p \|, \| x_n - Tx_n \|, \| x_n - p \| + \| x_n - Tx_n \| + \| Tp - Tx_n \|, \| p - Tx_n \| + \| x_n - Tx_n \| + \| Tx_n - Tp \| \}$$

Thus we obtain, by taking the limit as $n \to \infty$,

$$\| Tx_n - Tp \| \to 0$$

At least

$$\| P - Tp \| \le \| p - Tx_n \| + \| Tx_n - Tp \| \to 0$$

This means

$$p = Tp.$$

Definition 2: A mapping T : $C \to C$ is called strictly pseuiocontractive if for some k, 0 ≤ k < 1, and all x, y, c, where X be a normed linear space and C be an non-empty subset of X.

$$\| Tx - Ty \|^2 \le \| x - y \|^2 + k \| (I-T)x - (I-T)y \|^2.$$

Definition 2.1: T is called pseuiocontractive if for all x,y | | C,

$$\| Tx - Ty \|^2 \le \| x - y \|^2 + \| (I-T)x - (I-T)y \|^2.$$

Definition 2.2: T is said to satisfy the condition (T) if for all x | | C and y | | F (T)

$$\| Tx - y \| \le \| x - y \|.$$

It is clear that any strictly pseuiocontractive mapping is hemicontractive, any mapping satisfying condition (T) is demicontractive and a demicontractive mapping is hemicontractive but not conversely.

Theorem 3: Let a mapping T : C $\to$ C satisfies condition (T). Suppose F (T) is non-empty and $\sum \alpha_n \beta_n$ diverges and $\beta_n \to \beta < 1$ Then $\lim \| x_n - Tx_n \| = 0$ for each x_O | | | |C | | | |where X_{n+1} is defined as in the Ishikawa scheme.

Proof: By using the condition (T), the mapping T is demicontractive for any constant K. We get for any x, y and z in H (Hillbert Space) and a real number.

$$\| \lambda x + (1-\lambda)y - z \|^2 = \| x - z \|^2 + (1-\lambda) \| y - z \|^2 - (1-\lambda) \| x - y \|^2$$

Therefore for p □ F (T) and each integer

$$n, 0 \le \| x_{n+1} - p \|^2 = \| \alpha_n Ty_n + (1-\alpha_n)x_n - p \|^2$$

$$= (1-\alpha_n) \| x_n - p \|^2 + \alpha_n \| Ty_n - p \|^2 - \alpha_n (1-\alpha_n) \| x_n - Ty_n \|^2$$

$$\le (1-\alpha_n) \| x_n - p \|^2 + \alpha_n \| Ty_n - p \|^2$$

By using condition (T) we get

$$\| x_{n+1} - p \|^2 \le (1-\alpha_n) \| x_n - p \|^2 + \alpha_n \| y_n - p \|^2.$$

By using demi contractiveness of T and definition of y_n we get

$$\| y_n - p \|^2 \le \| x_n - p \|^2 - \beta_n (1 - \beta_n - k) \| x_n - Tx_n \|^2$$

Hence,

$$0 \le \| x_{n+1} - p \|^2 \le (1-\alpha_n) \| x_n - p \|^2 + \alpha_n \| x_n - p \|^2 - \alpha_n \beta_n (1 - \beta_n - k) \| x_n - Tx_n \|^2 \quad \ldots(3.1)$$

$$= \| x_n - p \|^2 - \alpha_n \beta_n (1 - \beta_n - k) \| x_n - Tx_n \|^2.$$

By induction, we obtain

$$\| x_0 - p \|^2 - \sum_{i=0}^{n} \alpha_i \beta_i (1 - \beta_i - k) \| x_i - Tx_i \|^2 \geq 0$$

Therefore

$$\sum_{n=0}^{\infty} \alpha_n \beta_n (1 - \beta_n - k) \| x_n - Tx_n \|^2 \leq \| x_n - p \|^2 \qquad \ldots (3.2)$$

We note that

$$\beta_n (1 - \beta_n) \leq \beta_n$$

$$(0 \leq \beta_n \leq 1).$$

Let $\eta = 1 - \beta - k$. Then $\eta > 0$ and ...... an integar N s.t. $\beta + \eta/2 > \beta_n$ for all $n \geq N$.

Therefore

$$\left(1 - k - \beta - \eta/2\right) = \eta/2 < 1 - \beta_n - k.$$

Hence

$$\sum \alpha_n \beta_n (1 - \beta_n - k) \geq \eta/2 \sum \alpha_n \beta_n$$

which diverges.

Therefore $\sum \beta_n \alpha_n (1 - \beta_n - k)$ diverges. And from (2.4) we get

$$\underline{\lim} \| x_n - Tx_n \| = 0.$$

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