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A Study on Fixed Point Theory in Generalized Modular Metric Space

Sandeep Singh*

Government Post Graduation College, Ambala, Haryana

Abstract – In this paper, considering both a modular metric space and a generalized metric space in the feeling of Jleli and Samet, we present another idea of generalized modular metric space. At that point we present a few examples demonstrating that the generalized modular metric space incorporates some sort of metric structures. At long last, we give some fixed point results to both withdrawal and quasicontraction type mappings on generalized modular metric spaces.

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INTRODUCTION

In 1990, the fixed point theory in modular function spaces was started by Khamsi, Kozlowski, and Reich (1990).Modular function spaces are an uncommon instance of the theory of modular vector spaces presented by Nakano (1950). Modular metric spaces were presented in Chistyakov, V.V.(2010). Fixed point theory in modular metric spaces was contemplated by Abdou and Khamsi (2013). Their methodology was generally not quite the same as the one concentrated in Chistyakov, V.V.(2010). In this paper, we pursue a similar methodology as the one utilized in Abdou, A.A.N., Khamsi, M.A.(2013).

Speculations of standard metric spaces are fascinating in light of the fact that they take into account some profound comprehension of the traditional outcomes got in metric spaces. One has dependably to be cautious when thinking of another speculation. For example, on the off chance that we loosen up the triangle disparity, a portion of the traditional known certainties in metric spaces may wind up difficult to acquire. This is the situation with the generalized metric separation presented by Jleli and Samet in Jleli, M., Samet, B.(2015). The creators demonstrated that this speculation envelops metric spaces, b-metric spaces, disjoined metric spaces, and modular vector spaces. In this paper, considering both a modular metric space and a generalized metric space in the feeling of Jleli and Samet (2015), we present another idea of generalized modular metric space.

Definition : Let *X* be an abstract set. A function $D:(0,\infty)\times X\times X\to[0,\infty]$ is said to be a regular generalized modular metric *(GMM)* on *X* if it satisfies the following three axioms:

(GMM_1) If $D_\lambda(x,y)=0$ for some $\lambda>0,$ then *x-y* for all $x,y\in X;$ (GMM_2) $D_\lambda(x,y)=D_\lambda(y,x)$ for all $\lambda>0$ and $x,y\in X;$ *(GMM$_3$)* There exists C > 0 such that, if $(x,y)\in X\times X,\{x_n\}\subset X$ with $\lim_{n\to\infty}D_\lambda(x_n,x)=$0 for some $\lambda>0,$ then $D_\lambda(x,y)\le C\limsup\limits_{n\to\infty}D_\lambda(x_n,y).$

The pair *(X,D)* is said to be a generalized modular metric space *(GMMS).*

It is easy to check that if there exist *x,y*$\in$*X* such that there exists $\{x_n\}\subset X$ with $\lim_{n\to\infty}D_\lambda(x_n,x)=0$ for some $\lambda>0,$ and $D_\lambda(x,y)<\infty,$ then we must have $C\ge 1.$ In fact, throughout this work, we assume $C\ge 1.$ Let *D* be a *GMM* on *X.* Fix $x_0\in X.$ The sets

$$\begin{cases} X_D=X_D(x_0)=\{x\in X:D_\lambda(x,x_0)\to 0 \text{ as } \lambda\to\infty\} \\ X_D^\star=\{x\in X:\exists\lambda=\lambda(x)>0 \text{ such that } D_\lambda(x,x_0)<\infty\}\end{cases}$$

are called *generalized modular sets.* Next, we give some examples that inspired our definition of a *GMMS.*

Example : (Modular vector spaces *(MVS)* [13]) Let A be a linear vector space over the field R. A function $\rho:X\to[0,\infty]$ is called regular modular if the following hold:

1. $\rho(x)=0$ if and only if *x*= 0,
2. $\rho(\alpha x)=\rho(x)$ if $|\alpha|=1,$
3. $\rho(\alpha x+(1-\alpha)y)\le\rho(x)+\rho(y)$ for any $\alpha\in[0,1],$ for any $x,y\in X$.

Let ρ be regular modular defined on a vector space X. The set $X_\rho = \left\{x \in X; \lim_{\alpha\to 0} \rho(\alpha x) = 0\right\}$ is called a *MVS.* Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ρ and $x \in X_\rho$. If $\lim_{n\to\infty} \rho(x_n - x) = 0$, then $\{x_n\}_{n\in\mathbb{N}}$ is said to ρ-converge to *x.* ρ is said to satisfy the Δ_2-condition if there exists $K \neq 0$ such that $\rho(2x) \leq K\rho(x)$ for any $x \in X_\rho$. Moreover, ρ-is said to satisfy the Fatou property (FP) if $\rho(x-y) \leq \liminf_{n\to\infty} \rho(x_n - y)$, whenever $\{x_n\}$ ρ-converges to x for any $x, y, x_n \in X_\rho$. Next, we show that a MVS may be embedded with a GMM structure. Indeed, let (X,ρ) be a *MVS.* Define $D : (0,+\infty) \times X \times X \to [0,+\infty]$ by

$$D_\lambda(x,y) = \rho\left(\frac{x-y}{\lambda}\right).$$

Then the following hold:

1. If $D_\lambda(x,y) = 0$ for some $\lambda > 0$ and any $x, y \in X$, then x = y,

2. $D_\lambda(x,y) = D_\lambda(y,x)$ for any $\lambda > 0$ and $x, y \in X$;

3. If ρ-satisfies the FP, then for any $\lambda > 0$ and {*„) such that $\{x_n/\lambda\}$ ρ-converges to x/λ, we have

$$\rho\left(\frac{x-y}{\lambda}\right) \leq \liminf_{n\to\infty} \rho\left(\frac{x_n - y}{\lambda}\right) \leq \limsup_{n\to\infty} \rho\left(\frac{x_n - y}{\lambda}\right),$$

which implies

$$D_\lambda(x,y) \leq \liminf_{n\to\infty} D_\lambda(x_n, y) \leq \limsup_{n\to\infty} D_\lambda(x_n, y)$$

for any $x, y, x_n \in X_\rho$

Hence, (X,D) fulfills every one of the properties of Definition as asserted. Note that the steady C which shows up in the property (GMMs) is equivalent to 1 given the FP is fulfilled by ρ.

SOME FIXED POINTS THEOREMS IN MODULAR SPACES

In this area we talk about the presence of fixed points for mappings which are nonresponsive or contractive in the modular sense. Positively, one can likewise consider mappings which are contractive regarding the F-standard prompted by the modular. It is worth to make reference to that, for the most part talking, there is no common connection between the two sorts of nonexpansiveness. By and by we might want to underline our rationality that every one of the outcomes communicated as far as modulars are progressively helpful as in their presumptions are a lot simpler to check.

Definition. Let *C* be a subset of a modular space and let $T : C \to C$ be an arbitrary mapping. L_ρ

1. *T* is a ρ-contraction if there exists $\lambda < 1$ such that $\rho(T(f) - T(g)) \leq \lambda\, \rho(f-g)$ for all $f, g \in C$.

2. *T* is said to be ρ-nonexpansive if $\rho(T(f) - T(g)) \leq \rho(f-g)$ for all $f, g \in C$

3. $f \in C$ is said to be a fixed point of *T* if *T(f)* = *f.* The fixed point set of *T* will be denoted *Fix(T).*

C will be said to have the fixed point property assuming each ρ- nonexpansive self-map defined on C has a fixed point.

A simple to banach compression standard, can be expressed as pursues.

Theorem . Leave C alone ρ- complete ρ- limited subset of L_ρ and $T : C \to C$ be a ρ- severe compression. At that point T has a unique fixed point $z \in C$ ". Additionally z is the ρ- furthest reaches of the emphasize of any point in C under the activity of T.

Review that a subset D of L_ρ is said to be p-finished assuming each ρ- Cauchy grouping from D is joined in D.

We may loosen up the suspicion with respect to the boundedness of C and expect there exists a limited circle. For this situation, the uniqueness of the fixed point is dropped and supplanted by on the off chance that f and g are two fixed points of T with the end goal that $\rho(f-g) < \infty$, at that point $f = g$

NORMAL STRUCTURE IN MODULAR SPACES

The idea of normal structure was presented by Brodskii and Milmann for the instance of linear normed spaces. Kirk was the first to interface this idea to presence of fixed point of nonexpansive mappings. There were a few endeavors to sum up the idea of normal structure to metric spaces and progressively theoretical sets. In this segment we define normal structure for function modulars.

Definition 1. Let *B* be a ρ-bounded subset of L_ρ

1. By the ρ-diameter of *B*, we will understand the number $\delta_\rho(B) = \sup\{\rho(f-g); f, g \in B\}$.

2. The quantity $r_\rho(f, B) = \sup\{\rho(f - g); g \in B\}$ will be called the ρ- Chebyshev radius of *B* with respect to *f.*

3. The ρ-Chebyshev radius of *B* is defined by $R_\rho(B) = \inf\{r_\rho(f, B); f \in$ *B}.*

4. The ρ-Chebyshev center of *B* is defined as the set $C_\rho(B) = \{f \in B; r_\rho(f, B) = R_\rho(B)\}$.

Note that $R_\rho(B) \le r_\rho(f, B) \le \delta_\rho(B)$ of or all $f \in B$ and observe that there is no reason, in general, for $C_\rho(B)$ to be nonempty.

Definition 2. Let *B* be a ρ-bounded subset of L_ρ.

(a) We say that / is a ρ-diametral point of *B* if $r_\rho(f, B) = \delta_\rho(B)$

(b) The set *B* is called ρ-diametral if every $f \in B$ is a ρ-diametral point of *B.*

(c) A sequence $\{f_n\}$ from L_ρ is called a ρ-diametral sequence if there exists c > 0 such that $\delta_\rho(f_n) \le c$ and $\lim_{n\to\infty} dist_\rho(f_{n+1}, conv(f_1, \ldots, f_n)) = c.$

where $dist_\rho(f, A) = \inf\{\rho(f - g); g \in A\}$ and

$$conv(f_1, .., f_n) = \Big\{\sum_{i=1}^{n} \alpha_i f_i \; ; \alpha_i \ge 0 \text{ and } \textstyle\sum_i \alpha_i = 1\Big\}.$$

Let us observe that $dist_\rho(f_{n+1}, conv(f_1, .., f_n)) \le nc,$ while in the norm case this distance can be estimated by the number c itself.

Definition 3. Let *B* be a ρ-bounded subset of L_ρ.

1. We say that *A* is an admissible subset of *B* if

$$A = \bigcap_{i\in I} B_\rho(b_i, r_i) \cap B \,,$$

Where $b_i \in B$, $r_i \ge 0$ and / is an arbitrary index set.

2. If C is a subset of B, we let

$$co(C) = \bigcap_{f\in C} B_\rho(f, r_\rho(f, C)) \cap B \,.$$

3. B is said to have ρ- normal structure property assuming every ρ- acceptable subset An of B, not diminished to a solitary point, has a point which isn't ρ- diametral.

By A(B) we signify the group of every single acceptable subset of B. Note that in the event that B is ρ- bounded, at that point $B \in \mathcal{A}(B)$

The traditional confirmation of Kirk's fixed point theorem depends vigorously on a topological minimization supposition. Under our plan, we may accept a successive conservativeness which is anything but difficult to check in numerous down to earth cases. Subsequently we may ask when our consecutive smallness is produced from a topological compactness, i.e. given a set which is consecutively reduced (in our sense), is there a topology which makes it minimal? This issue as far as anyone is concerned is as yet open. Thusly, we will utilize a "build if confirmation" developed by Kirk rather than the established evidence dependent on Zorn and the compactness presumption. It is worth to make reference to that Kirk's fixed point theorem in modular spaces is the principal example where the construct if evidence was utilized.

FIXED POINT THEOREMS (FPT) IN GMMS

The accompanying definition is valuable to set new fixed point theory on GMMS.

Definition 1 Let (X_D, D) be a GMMS.

1. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be D-convergent to $x \in X_D$ if and only if $D_\lambda(x_n, x) \to 0,$ as $n \to \infty,$ for some $\lambda > 0.$

2. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be *D-Cauchy* if $D_\lambda(x_m, x_n) \to 0,$ as $m, n \to \infty,$ for some $\lambda > 0.$

3. A subset C of X_D is said to be D-closed if for any $\{x_n\}$ from C which D-converges to $x, x \in C.$

4. A subset C of X_D is said to be D-complete if for any *(x_n)* D-Cauchy sequence in C such that $\lim_{n,m\to\infty} D_\lambda(x_n, x_m) = 0$ for some λ, there exists a point $x \in C$ such that $\lim_{n,m\to\infty} D_\lambda(x_n, x) = 0.$

4. A subset C of X_D is said to be D-bounded if, for some $\lambda > 0,$ we have $\delta_{D,\lambda}(C) = \sup\{D_\lambda(x, y); x, y \in C\} < \infty.$

In general, if $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for some $\lambda > 0,$ then we may not have $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for all $\lambda > 0.$ Therefore, as it is done in modular function spaces, we will say that *D* satisfies Δ_2-condition if and only if $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for some $\lambda > 0$ implies $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for all $\lambda > 0.$

Another question that comes into this setting is the concept of D-limit and its uniqueness.

Proposition 2.1 *Let {Xd, D) be a* GMMS. *Let [x_n) be a sequence in Xd. Let* $(x, y) \in X_D \times X_D$ *such that*

$D_\lambda(x_n,x) \to 0$ and $D_\lambda(x_n,y) \to 0$ *as* $n \to \infty$ *for some* $\lambda > 0.$ *Then x-y.*

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Corresponding Author

Sandeep Singh*

Government Post Graduation College, Ambala, Haryana