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On Common Fixed Points Results in Cone Rectangular Metric Spaces

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Abstract – The present paper provides generalization of the fixed point theorem on complete cone rectangular metric space. Here we have proved fixed point theorem by taking f as self mapping from X into itself.

Key Word: Self Mapping, Banach Space, Cone Metric Space, Convergent.

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1. INTRODUCTION

Recently, Guang and Xian [4] introduced a new metric space known as a cone metric space. Subsequently Azam et al. [1] have given the idea of cone rectangular metric space. Here, we will introduce some new results on cone rectangular metric space.

Let E is a real Banch space and P is a subset of E.P. is called a cone if and only if it satisfies the following conditions.

(i) P is closed, non-empty and $P \neq \{0\}$

(ii) $a, b \in R$ and $a, b \geq 0, u, v \in P \Rightarrow au + bv \in P$

(iii) $u \in P$ and $-u \in P \Rightarrow u = 0$

Definition – 1 : Let X be a nonempty set. Suppose that d is a mapping from X × X $\rightarrow$ E, satisfies :

(i) $d(x, y) > 0, \forall x, y \in X$

(ii) $d(x, y) = 0$ if and only if $x = y$

(iii) $d(x, y) = d(y, x), \forall x, y \in X$

(iv) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$

Then d is called a cone metric on X and (X, d) is called cone metric space.

Definition-2 : Let x be a nomempty set. Suppose the mapping d : x × x $\rightarrow$ E, satisfies.

(i) $0 \leq d(x, y); \forall x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$

(ii) $d(x, y) = d(y, x), \forall x, y \in X.$

(iii) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y), \forall x, y \in X$

and for all distinct point $w, z \in X - \{x, y\}$ [rectangular property]

Then d is called a cone rectangular metric on X, and (X, d) is called a cone rectangular metric space.

Definition -3 : Let $\{x_n\}$ be a sequence in (X, d) and x $\in$ (X, d). If for every c $\in$ E, with 0 < < c there is n_0 $\in$ N such that for all n > n_o, d (x_n, x) << c, then $\{x_0\}$ is said to be convergent and converges to x.

i.e. $\lim_{n \to \infty} x_n = x$

Definition-4 : A sequence (x_n) is said to be Cauchy in X if for c $\in$ E with 0 < < c there is n_0 $\in$ N such that for all $n, m > n_0, d(x_n, x_m) << c$ then $\{x_n\}$ is called Cauchy sequence.

Definition – 5: A cone rectangular metric space is said to be complete cone rectangular metric space if evry Canchy sequence is X is convergent.

MAIN RESULT:

Theorem (1.1) : Let (X, d) is a complete cone rectangular metric space and P is a normal cone with normal constant K. Let f is self-mapping from X into itself satisfying.

$$d(f_x, f_y) \le \alpha\left\{\frac{d(x,y)+d(x,fx)+d(fx,y)}{2}+d(y,fy)\right\}$$

$$\forall x, y \in X,\ \alpha \in [0,1) \text{ and } 0 < \frac{\alpha}{1-\alpha} < 1$$

Then f has an unique fixed point in X.

Proof : Let $x_0 \in X$ be an arbitrary point in X. Let us take a sequence $\{x_0\}$ in X

such that

$$x_{n+1} = fx_n = f_{x_0}^{n+1}, n \in N \cup \{0\}$$

Now substituting $x = x_0$ and $y = x_1$ in inequality we obtain

$$d(x_0, fx_1) = d(x_1, x_2)$$

$$\le \alpha\left\{\frac{d(x_0,x_1)+d(x_0,fx_0)+d(fx_0,x_1)}{2}+d(x_1,fx_1)\right\}$$

$$\le \alpha\left\{\frac{d(x_0,x_1)+d(x_0,x_0)+d(x_1,x_1)}{2}+d(x_1,x_1)\right\}$$

$$\Rightarrow d(x_1,x_2) \le \alpha(\alpha\{d(x_0x_1)+d(x_1,x_2)\}$$

$$\Rightarrow d(x_1,x_2) \le \frac{\alpha}{1-\alpha} d(x_0,x_1)$$

$$\Rightarrow d(x_1,x_2) \le hd(x_0x_1) \qquad (1.1)$$

Where $0 < h = \frac{\alpha}{1-\alpha} < 1$

Again for $x = x_1, y = x_2$ we have

$$d(x_2,x_3) = d(fx_1, fx_2)$$

$$\le \left\{\frac{d(x_1,x_2)+d(x_1,fx_1)+d(fx_1,x_2)}{2}+d(x_2,fx_2\right\}$$

$$\Rightarrow d(x_2,x_3) \le \alpha\left\{\frac{d(x_1,x_2)+d(x_1,x_2)+d(x_2,x_2)}{2}+d(x_2,x_3)\right\}$$

$$\Rightarrow d(x_2,x_3) \le \frac{\alpha}{1-\alpha} d(x_1,x_2) \le h\, d(x_1,x_2)$$

$$\Rightarrow d(x_2,x_3) < h^2 d(x_0x_1) \qquad (1.2)$$

Thus in general we have for positive integer n

$$d(x_n,x_{n+1}) \le h^n d(x_0,x_1) \qquad (1.3)$$

Now for n > m and applying the Lemma [3, 5] we have

$$d(x_n,x_m) \le \left(h^{n-1}+h^{n-2}-h^{n-3}+........+h^m\right)d(x_0,x_1)$$

$$\le \frac{h^m}{1-h} d(x_0,x_1) \qquad (1.4)$$

Taking the normality of Cone. (1.4) gives

$$\| d(x_n,x_m) \| \le K\frac{h^m}{1-h} \| d(x_0,x_1) \| \qquad (1.5)$$

Which yields

$$\| d(x_n,x_m) \| \Rightarrow 0$$

$$\Rightarrow d(x_n,x_m) \to 0 \qquad (1.6)$$

Now we will claim that our inequality satisfies the rectangular property for finding the fixed point in X. Because of this we will calculate the following results.

For $y \in X$ we have

$$d(fy,f^2y) \le \alpha\left\{\frac{d(y,fy)+d(y,fy)+d(fy,fy)}{2}+d(fy,f^2y)\right\} \qquad (1.7)$$

$$\Rightarrow d(fy,f^2) \le \frac{\alpha}{1-\alpha} d(y,fy) \le hd(y,fy)$$

Again

$$d(f^2y,f^3y) = d(ffy,ff^2y)$$

$$\le \alpha\left\{\frac{d(fy,f^2y)+d(fy,f^2y)+d(f^2y,f^2y)}{2}+d(f^2y,f^3y)\right\}$$

$$\Rightarrow d(f^2y,f^3y) \le h^2 d(y,fy) \qquad (1.8)$$

Thus in general for positive integer n

$$d(f^n y,f^{n+1}y) \le h^n d(y,fy) \qquad (1.9)$$

Now from rectangular property we have for $y \in X$

$$d\left(fy,f^4y\right) < d(fy,f^2y)+d(f^2y,f^3y)+d(f^3y,f^4y)$$

$$\le hd(y,fy)+h^2d(y,fy)+h^3d(y,fy) \le \sum_{i=1}^{3} h^i d(y,fy)$$

Similarly

$$d(f^2y,f^5y) < d(f^2y,f^3y)+d(f^3y)+d(f^3y,f^4y)+d(f^4y,f^5y)$$

$$\le h^2 d(y,fy)+h^3 d(y,fy)+h^4 d(y,fy)$$

$$\le \sum_{i=2}^{4} h^i d(y,fy)$$

Thus in general for n > m and from Lemma [3, 5]

$$d(f^n y, f^m y) \le (h^{n-1}+h^{n-2}+h^{n-3}+.....h^m) d(y,fy)$$

$$\le \frac{h^m}{1-h} d(y,fy)$$

Now for $x_0 = y \in X$

$$d(f^n x_0, f^m x_0) \le \frac{h^m}{1-h} d(x_0, fx_0)$$

$$\Rightarrow d(x_n, x_m) \le \frac{h^m}{1-h} d(x_0, x_1)$$

Applying the normality of cone we obtain.,

$$\| d(x_0, x_m) \| \le \frac{h^m}{1-h} K \| d(x_0, x_1) \|$$

Which implies that $\| d(x_n, x_m) \| \to 0$

$$\Rightarrow d(x_n, x_m) \to 0 \text{ as } n, m \to \infty \qquad (1.10)$$

Now we can conclude from (1.6) and (1.10) that (x_n) is a Cauchy sequence in X. As x is complete cone rectangular metric space then there exists a point x in (X, d) such that

$$x_n \to x \text{ as } n \to \infty$$

Now $d(x_n, fx_n) = d(fx_{n-1}, fx_n)$

$$\le \alpha \left\{ \frac{d(x_{n-1}, x_n) + d(x_{n-1}, fx_{n-1}) + d(fx_{n-1}, x_n)}{2} + d(x_n, fx_n) \right\}$$

Letting $n \to \infty$ we get

$$d(x, fx) \le \alpha \left\{ \frac{d(x,x) + d(x,fx) + d(fx,x)}{2} + d(x,fx) \right\}$$

$$\Rightarrow d(x,fx) \le \alpha \{ d(x,fx) + d(x,fx) \}$$

$$\Rightarrow (1-2\alpha) d(x,fx) \le 0$$

Now applying the normality of cone we have,

$(1-2\alpha) \| d(x,fx) \| \le 0$ which is not possible as $d(x,y) \ge 0$ and $(1-2\alpha) > 0$ and hence $d(x,fx) = 0 \Rightarrow fx = x$

Hence x is a common fixed point off in x.

Now we will prove that x is unique. If possible let there exists another fixed point x' of f in (X,d)..

$$fx' = x'$$

Then $d(x,x') = d(fx, fx')$

$$\le \alpha \left\{ \frac{d(x,x') + d(x,fx) + d(fx,x)}{2} + d(x',fx) \right\}$$

$$\Rightarrow d(x,x') \le \alpha \left\{ \frac{d(x,x') + d(x,x')}{2} + d(x',x') \right\}$$

$$\Rightarrow d(x,x') \le \alpha d(x,x')$$

$$(1-\propto) d(x,x') \le 0$$

Again a contradiction and hence

$$d(x,x') = 0$$

$$\Rightarrow x = x'$$

Hence x is an unique common fixed point off in X. This completes the proof of the theorem.

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