

Chaotic Study of Dynamical Systems: A Survey

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Abstract – Nonlinear dynamics and chaotic systems are of massive interest to mathematicians, physicists and engineers in nineteenth century. In this paper, it has been tried to give a brief survey on the behavior of the quadratic map, logistic map and tent map using different iteration. In this paper i also discuss the behavior of logistic map using bifurcation diagram. Also, here I also discuss the range of the stability of the logistic map using Picard, Maan iterations. Also, in this paper I discuss the most useful algorithm Parrondo's Paradox which give "Chaos + Chaos = Order". Also I discuss the behavior of tent map. The range of the parameter is one for which the tent map has stable behavior.

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1. INTRODUCTION

In mathematics, a dynamical system is a system in which a function describes the time dependence of a point in a geometrical space. It is the study of iteration of maps over a time period [4]. Nonlinear dynamics studies systems governed by equations more complex than the linear, $ax+b$ form. Nonlinear systems, such as the weather or neurons, often appear chaotic, unpredictable and yet their behavior is not random.

Chaos theory describes the behavior of certain dynamical systems – that is, systems whose state evolves with time – that may exhibit dynamics that are highly sensitive to initial conditions. The butterfly effect describes how a small change in one state of a deterministic nonlinear system can result in large differences in a later state, e.g. a butterfly flapping its wings in Brazil can cause a hurricane in Texas [6].

Chaotic behavior exists in many natural systems, such as weather and climate [5, 12]. It also occurs spontaneously in some systems with artificial components, such as road traffic [3]. This behavior can be studied through analysis of a chaotic mathematical model, or through analytical techniques such as recurrence plots and Poincaré maps.

2. CHAOS THEORY

Chaos: When the present determines the future, but the approximate present does not approximately determine the future.

In common usage, "chaos" means "a state of disorder". Although no universally accepted mathematical definition of chaos exists, a commonly used definition originally formulated by Robert L. Devaney [3] says that, to classify a dynamical system as chaotic, it must have these properties:

1. it must be sensitive to initial conditions
2. it must be topologically mixing
3. it must have dense periodic orbits

Definition 2.1. Let D be a subset of a metric space with metric d . The function $f : D \rightarrow D$ exhibits sensitive dependence on initial conditions if there exists a $\delta > 0$ such that for any x in D and any $\epsilon > 0$, there is a y in D and a natural number n such that $d[x,y] < \epsilon$ and $d[f^n(x), f^n(y)] > \delta$.

Definition 2.2. Let D be a subset of a metric space. The function $f: D \rightarrow D$ is topologically transitive on D if for any two points x and y in D and any $\epsilon > 0$, there is z in D such that $d[z,x] > \epsilon$ and $d[f^n(z), y] > \epsilon$ for some n .

3. QUADRATIC MAP

A quadratic map is a quadratic recurrence equation of the form

$$x_{n+1} = a_2 x_n^2 + a_1 x_n + a_0.$$

where $a_0, a_1, a_2 \in \mathbb{R}$, and $a_2 \neq 0$. Since all the quadratic maps are topologically conjugate to each other.

Now, we consider the quadratic map $Q_c(x) = x^2 + c$ and discuss their dynamics for different values of c . First, we compute the fixed points of $Q_c(x)$. Fixed points are obtained by solving the quadratic equation

$$x^2 + c = x$$

which yields the roots:

$$p_+ = \frac{1}{2}(1 + \sqrt{1-4c})$$

$$p_- = \frac{1}{2}(1 - \sqrt{1-4c})$$

Clearly, p_+ and p_- are real if and only if $1 - 4c \geq 0$. Now, we have the following cases:

Case 1. when $c > 1/4$, $Q_c(x)$ has no fixed points.

Case 2. $Q_c(x)$ has only one fixed point when $c = 1/4$. All orbit tends to infinity when $c > 1/4$. And at $c = 1/4$, $|Q'_c(x)| = 1$ implies that $p_+ = p_-$ is neutral fixed point.

Case 3. When $c < 1/4$, $Q_c(x)$ has two fixed points p_+ and p_- . The fixed point p_+ is always repelling.

- (a) If $-3/4 < c < 1/4$, p_- is attracting fixed point.
- (b) If $c = -3/4$, p_- is neutral fixed point.
- (c) If $c < -3/4$, p_- is repelling fixed point.

We observe that for any $c \leq 1/4$, all the interesting dynamics occurs in the interval $-p_+ < x < p_+$. Also $Q_c(-p_+) = p_+$, i.e. $-p_+$ is an eventually fixed point. For $-3/4 < c < 1/4$, all orbits in the interval $(-p_+, p_+)$ tends to the attracting fixed point p_- .

Now we discuss the dynamics of $Q_c(x)$ when c decreases below $-3/4$. The nature of p_- is changes from attracting to repelling as c decreases from $-3/4$. Also, there are no other cycles when $c > -3/4$ but cycle of period 2 appears when $c < -3/4$. The periodic points of periodic 2 of $Q_c(x)$ are the fixed points of $Q_c^2(x)$. These periodic points of periodic 2 of $Q_c(x)$ are given by

$$q_{\pm} = \frac{1}{2}(-1 \pm \sqrt{-4c-3})$$

From above we see that the value of q_{\pm} also depends on c and they are real only if $c \leq -3/4$. Here

we have bifurcation called a period-doubling bifurcation.

Case4. When $c \leq -3/4$,

- (a) $c = -3/4$, $Q_c(x)$ has a neutral fixed point $p_- = q_{\pm}$ and no 2 cycles.
- (b) $-5/4 < c < -3/4$, $Q_c(x)$ has repelling fixed points at p_{\pm} and an attracting 2-cycle at q_{\pm} .

Case 5. Now we discuss the dynamics of $Q_c(x)$ case when $c = -2$.

The interesting dynamics of $Q_c(x)$ take place in the interval $-p_+ < x < p_+$ where p_+ is repelling fixed point when $p_+ > 0$. For $c = -2$, we have $p_+ = 2$. Let $I = [-2, 2]$, the graph of $Q_{-2}(x)$ on I is shown in the Fig.

3.1. Note that, Q_{-2} is increasing on the interval $[0, 2]$ and takes subinterval into the entire interval I in one to one fashion. Similarly, Q_{-2} is decreasing on the subinterval $[-2, 0]$ and takes this interval on to I in one to one fashion. Thus, every point in I has exactly two pre images in I one on $[-2, 0]$ and one in $[0, 2]$. Thus, we can say that Q_{-2} stretching and folding the interval I and then mapping it back over itself twice as depicted in Fig 3.2. Similarly, the graph of Q_{-2}^2 and

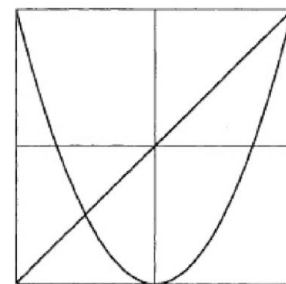


Fig. 3.1 The graph of Q_{-2} on $[-2, -2]$

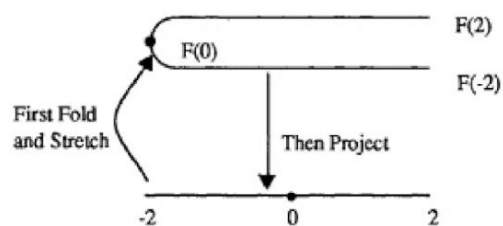


Fig. 3.2

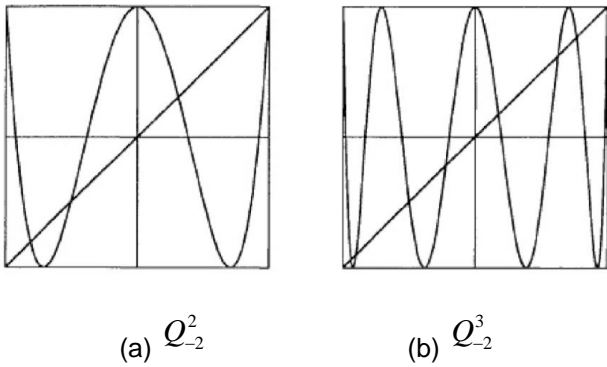


Fig. 3.3 The graph of Q_{-2}^2 and Q_{-2}^3 on $[-2, 2]$

Q_{-2}^3 as shown in Fig. 3.3. It is clear from the Fig 3.2 that Q_{-2}^2 has four fixed points and Q_{-2}^3 has eight fixed points. Continuing in this manner, we see that the Q_{-2}^n has 2^n fixed points [3].

Theorem [3]. The function Q_{-2} has at least 2^n periodic points of period n in the interval $-2 \leq x \leq 2$.

Case 6. When $c < -2$, in this section we discuss the variety of orbits of $Q_c(x)$ when $c < -2$. Nearly all the orbits of $Q_c(x)$ tends to infinity when $c < -2$, so it appears that the dynamics in this case are relatively simple. The Fig. 3.4 show the graph of $Q_c(x)$ for a c -value less than -2 .

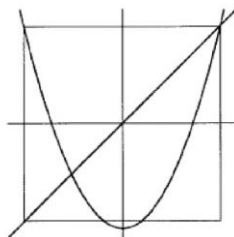


Fig. 3.4 The graph of Q_c for $c < -2$

The graphical analysis show that unlike the graph of Q_{-2} , in graph of $Q_c(x)$ for $c < -2$ there is an open interval in I containing 0 that is mapped outside of I by Q_c . In particular, the orbit of any point in this interval escape from I and so tends to infinity. Let's call this interval A_1 . A_1 is the set of points that escape from I after just one iteration of Q_c . Also, any orbit that eventually leaves I must tend to infinity, so we understand the fate of all the orbits. So, it remains

to understand the fate of orbits that never escape from I . Let us denote the set of points by Λ . That is,

$$\Lambda = \{x \in I \mid Q_c^n(x) \in I \text{ for all } n\}$$

Let A_n denotes the set of all points in I whose orbits leaves I after exactly n iterations. Since each element in I has exactly two pre images in I , thus the set A_n consists of exactly 2^{n-1} open intervals. If a point has an orbit that eventually escape from I , then this point must lie in some A_n for $n \in \mathbb{N}$. Hence the complement of Λ in I is just the complement of $\bigcup_{n=1}^{\infty} A_n$.

Now, to construct Λ , we first remove A_1 from the interval I , that leaves two close intervals behind. Then we remove A_2 from I , that leaves four close intervals behind. Continuing, when we remove A_n , we are left with 2^n closed intervals. The set Λ is

what is left after we remove $\bigcup_{n=1}^{\infty} A_n$. This set is called a Cantor set [3]. Fig 3.5 shows the full orbit diagram of Q_c .

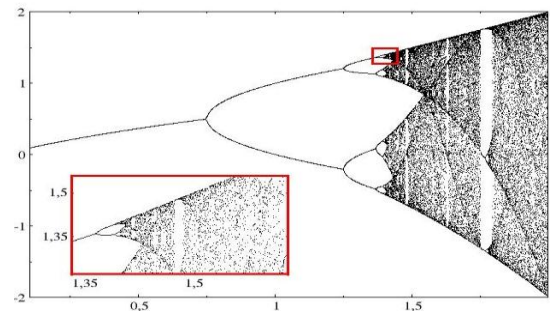


Fig. 3.5. Orbit diagram of $Q_c(x) = x^2 + c$ and a magnification.

4. LOGISTIC MAP

The logistic map is a polynomial mapping of degree 2. It was first created by Pierre Franois Verhulst [10]. Mathematically, the logistic map is written

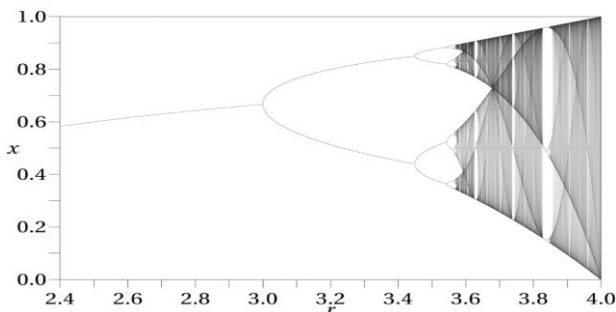
$$f_r(x) = x_{n+1} = rx_n(1 - x_n)$$

where x_n is a number between zero and one and r is called a parameter and family is called a parameterized family. In many applications, the

map is a model for the dynamics of a population, and x_n is the population of the n th generation. The fixed points for this map are at $x = 0$ and $P_r = 1 - 1/r$.

Also 1 and $\frac{1}{r}$ are fixed points, since $f_r(1) = 0$ and $f_r(\frac{1}{r}) = P_r$. The derivative of $f(x)$ is $f'(x) = r - 2rx$, and the equilibrium point is stable if $|f'(P_r)| < 1$. Since $|f'(1 - 1/r)| = |2 - r|$, this point is stable if $1 < r < 3$, and unstable otherwise. On the other hand, when $0 < r < 1$, the other equilibrium point $x = 0$ is stable since $|f'(0)| = |r|$.

The image given below shows the amplitude and frequency content of some logistic map iterates for parameter values ranging from 2.4 to 4.



By varying the parameter r , the following behavior is observed:

- With r between 0 and 1, the population will eventually die, independent of the initial population.
- With r between 1 and 2, the population will quickly approach the value $\frac{r-1}{r}$, independent of the initial population.
- With r between 2 and 3, the population will also eventually approach the same value $\frac{r-1}{r}$, but first will fluctuate around that value for some time. The rate of convergence is linear, except for $r = 3$, when it is dramatically slow, less than linear.
- With r between 3 and $1 + \sqrt{6} \approx 3.44949$, from almost all initial conditions the population will approach permanent oscillations between two values. These two values are dependent on r .
- With r between 3.44949 and 3.54409 (approximately), from almost all initial conditions the population will approach permanent oscillations among four values, then 8, 16, 32, etc. The lengths of the

parameter intervals that yield oscillations of a given length decrease rapidly. This behavior is an example of a period-doubling cascade.

- At $r < 3.56995$ is the onset of chaos, at the end of the period-doubling cascade. From almost all initial conditions, we no longer see oscillations of finite period. Slight variations in the initial population yield dramatically different results over time, a prime characteristic of chaos.
- Beyond $r = 4$, almost all initial values eventually leave the interval $[0,1]$ and diverge.

Maan iteration in Logistic Function [7]: Let A be a subset of real numbers and $f:A \rightarrow A$. For $x_0 \in A$, construct a sequence $\{x_n\}$ in the following manner:

$$x_1 = \beta_1 f(x_0) + (1 - \beta_1)x_0;$$

$$x_2 = \beta_2 f(x_1) + (1 - \beta_2)x_1;$$

.....

$$x_n = \beta_n f(x_{n-1}) + (1 - \beta_n)x_{n-1};$$

Where $0 < \beta_n \leq 1$ and $\{\beta_n\}$ is convergent when its value is away from 0 [10]. For sake of simplicity we have considered $\beta_1, \beta_2, \beta_3, \dots, \beta_n = \beta$ in this paper.

Moreover, x_0 will be called Mann forward asymptotic or superior forward asymptotic to p if the sequence $\{x_n\}$ converges to p .

In the existing literature, one-step feedback process has been used for iterating logistic map $f(x) = rx(1 - x)$. In 2005, Rani and Kumar [10] iterated the logistic map under two-step feedback machine and found that in Mann orbit, logistic map is convergent for higher values of r . Also, Kumar and Rani [9] performed an experiment on logistic map to find its increased convergence range. The maximum value of r depends on the value of β for converging behavior.

By varying the parameter β , the following behavior is observed:

1. For $\beta = 0.9$, the Mann iterative procedure generates spiral for $r \leq 3.22, \forall x \in [0,1]$, which is convergent.
2. For $\beta = 0.8$, the value of $r \leq 3.5, \forall x \in [0,1]$ for which the Mann iterative procedure generates spiral, which is convergent. If we

make a small increment in r, the superior orbit does not converge. The Mann sequence of iterates creates a cycle when $r = 3.51$.

3. For $\beta = 0.7$, the value of r increases up to 3.85 for which the Mann iterative procedure generates a spiral which is convergent. Furthermore, if there is a small increment in the value of r then the iteration scheme is not convergent.
4. The value of r increases up to 4.33 when $\beta = 0.6$ for which the Mann iterative procedure is convergent for any initial value $x_0 \in [0,1]$. If the value of r is increase for the same value of β than iterative procedure is not convergent and after a limit it is not stable.
5. For $\beta = 0.5$, the range of r is (0,5] for which the Mann iterative scheme is convergent. For $\beta = 0.4$, the range of r is (0,6] and for $\beta = 0.3$, the range becomes (0,7.66] respectively for which the Mann iterative procedure is convergent.
6. For $\beta = 0.2$, the Mann iterative procedure is convergent for $0 > r \leq 10.66$. The logistic function shows unstable behavior for $r > 10.66$.
7. The Mann iterative procedure is convergent for $r \leq 17.33$ when $\beta = 0.1$ for any initial value $x_0 \in [0,1]$. Moreover, the value of r in increases up to 21 for $x_0 = 0.5$ when $\beta = 0.1$.

Consequently, we can say that the value of r increases continuously as the value of β decreases for which the Mann iterative scheme for the logistic map has suitable behavior. If we increase the value of r for the same value of β as discuss above then logistic map has chaotic behavior.

4.1 PARRONDO'S PARADOX

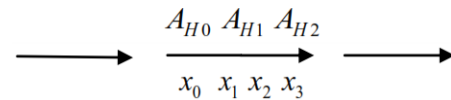
The Parrondo's paradox is coined by Spanish physicist Juan M. R. Parrondo in 1996 [13]. The main idea of this Paradox is that two simple games with negative gains when played individually and observed with positive gains by playing them alternatively in random or deterministic manner. There has been a lot of research on Parrano's games after the first published paper.

4.1.1 Parrondo's Paradox in the Superior logistic Map

In 2016, Mamta et al. [11] analyze the stability of the superior logistic map and study the dynamics of alternated logistic maps in superior orbit. It has been found that for higher values of r the periodic

alternation of two discrete dynamical systems are ordered whereas individually they are chaotic. In this paper, they studied stability analysis of the superior logistic map for two different cases one is for period-1 & second is period-2 cycle and also showed that for higher values of r the combination of two alternated chaotic superior logistic maps behave orderly.

In this paper, they assume two different discrete dynamics A1 and A2, and the alternation of the combination of the dynamics A1 and A2 is defined as



where H is a deterministic or random law which assigns value 1 or 2 to each number of the sequence {0, 1, 2...}, and $\{x_0, x_1, x_2, \dots\}$ are the values of variable x describing the physical system [11]. The two dynamics A1 and A2 are chaotic but the dynamics obtained by periodic alternation, $A_1A_2A_1A_2A_1A_2\dots = (A_1A_2)$ is ordered in well-defined sense and it is stated as "chaos + chaos = order" [1].

Initially, the work was done on two dynamics A1 and A2 defined by

$$\begin{array}{l} A_1 : x_{n+1} = \\ A_2 : x_{n+1} = \end{array} \begin{cases} x_n^2 + c_1, \\ x_n^2 + c_2, \end{cases}$$

and the range of c is $-2 \leq c \leq 0.25$ the invariant set under iteration of the real Mandelbrot map, i.e., set of initial condition yielding bounded orbits [11]. Alternated dynamics (A_1A_2) is defined as $(A_1A_2) : x_{n+1} = x_n^2 + c_1$ when n is odd, $x_{n+1} = x_n^2 + c_2$ when n is even,

where $x, c, c_1, c_2 \in \mathbb{R}$. As we know that $x_{n+1} = x_n^2 + c$ is topologically conjugate to the logistic map, $x_{n+1} = rx_n(1-x_n)$, it implies that "chaos + chaos = order" also arises in the logistic map [1].

Proposed Algorithm

Let $f_1(x) = r_1x(1-x)$ and $f_2(x) = r_2x(1-x)$ be the two logistic maps. When superior iterations are applied alternatively in $f_1(x)f_2(x)$, it becomes

$$f_1(x)f_2(x) = \begin{cases} x_{n+1} = \beta r_1 x_n (1-x_n) + (1-\beta)x_n & \text{if } n \text{ is odd;} \\ x_{n+1} = \beta r_2 x_n (1-x_n) + (1-\beta)x_n & \text{if } n \text{ is even,} \end{cases} \quad (4.1)$$

where $x, r_1, r_2 \in \mathbb{R}$.

4.1.2 Alternate Superior Logistic Map

Case 1

In 2005 Almedia et al. [1] takes two logistic maps with $r_1 = 3.574804938\dots$ and $r_2 = 4$ have been considered and when the alternation was applied it formed 8 periodic stable orbits. For the sake of analogy, they iterate the logistic maps $f_1(x), f_2(x)$ alternatively in SO (Eq. 4.1) for the above same values of r_1 and r_2 , and find the following results:

- ▶ For $1 \geq \beta \geq 0.85$, the logistic map $f_1(x) f_2(x)$ forms 8 periodic cycle.
- ▶ For $0.85 > \beta \geq 0.798$, it forms 4 periodic cycle.
- ▶ It was found that for $\beta < 0.798$, the logistic map $f_1(x), f_2(x)$ is 2-periodic.

The following Fig.4.1 show the stability graph at different values of β .

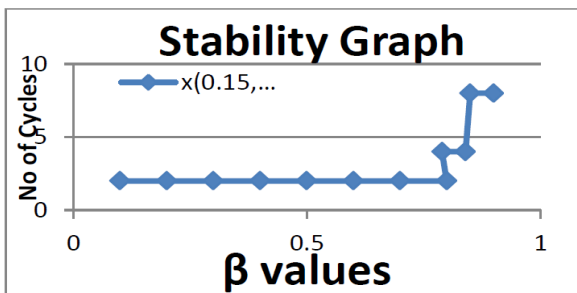


Fig. 4.1. Stability graph at different β -values

Hence, the system becomes more stable by using the proposed approach.

Case 2

Now, they consider the logistic map $f_1(x)f_2(x)$ for other values of r_1 and r_2 in SO (Eq. 4.1) and find following results:

- (i) There is no unstable situation in superior logistic map for $0 < \beta < 0.64$ for any choice of r_1 and r_2 .
- (ii) For $1 \geq \beta \geq 0.64$, they found some triplets of (β, r_1, r_2) for which “chaos + chaos = order”. Following are the three triplets in which the logistic maps are individually chaotic, but their alternate combination is ordered.
 - ▶ $(\beta = 0.95, r_1 = 3.705, r_2 = 4.105)$ is 8-periodic stable. See Fig.4.2

- ▶ $(\beta = 0.85, r_1 = 4.02, r_2 = 4.5126)$ is 8-periodic stable. See Fig. 4.3
- ▶ $(\beta = 0.75, r_1 = 4.4955, r_2 = 4.6155)$ is 6-periodic stable.

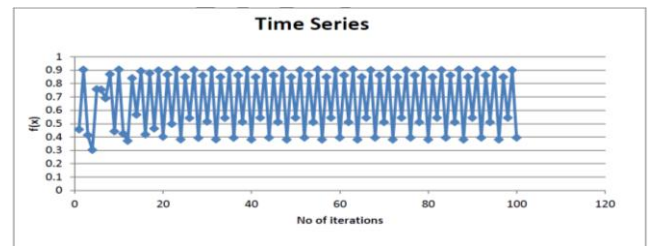


Fig. 4.2. 8-periodic cycle at $(x_0 = 0.15, \beta = 0.95, r_1 = 3.705$ and $r_2 = 4.105)$

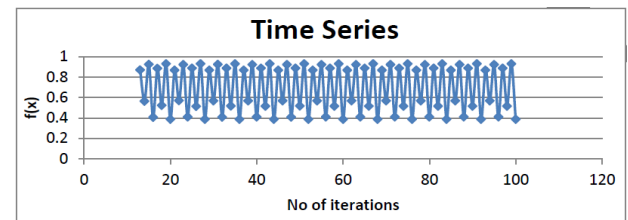


Fig. 4.3. 8-periodic cycle time series at $(x_0 = 0.15, \beta = 0.85, r_1 = 4.02$ and $r_2 = 4.5126)$

The following conclusions are drawn

- 1) For period-1, the superior logistic map is stable in superior orbit when $1 < r < (1 + 2/\beta)$.
- 2) For period-2, the logistic map may be stable for $r > (1 + 2/\beta)$.
- 3) Also, with the use of Parrondo’s paradox in the superior logistic map, in certain cases, two chaotic dynamical systems may result in an ordered system jointly.

4.1.3 Parrondo’s Paradox in the Noor Logistic Map

The logistic map $f(x) = r x (1 - x)$ shows various forms by choosing x between 0 and 1 and $r > 0$. The stability range of the logistic map in Noor orbit is more than the Picard, Mann & Ishikawa iteration. But still the chaotic situation occurs in the system. In this paper, Yadav et. al. studied the dynamics of alternated logistic maps in Noor orbit [14]. And it has been found that even the earlier chaotic situations of Noor orbit may become stable.

In this paper they assume two different discrete dynamical system $A_1: x_{n+1} = x_n^2 + c_1$, and $A_2: x_{n+1} = x_n^2 + c^2$,

and the alternation of the combination of the dynamics A1 and A2 is defined as

$$(A_1 A_2) : x_{n+1} = \begin{cases} x_n^2 + c_1 & \text{when } n \text{ is odd,} \\ x_n^2 + c_2 & \text{when } n \text{ is even,} \end{cases} \quad (4.2)$$

where $x, c, c_1, c_2 \in \mathbb{R}$. Since $x_{n+1} = x_n^2 + c$ topologically conjugate to the logistic map $x_{n+1} = rx_n(1-x_n)$, it implies that “chaos + chaos = order” also arises in the logistic map [1].

Proposed Algorithm: Let $f_1(x) = r_1 x(1-x)$ and $f_2(x) = r_2 x(1-x)$ be the two logistic maps. When Noor iteration are applied alternatively in $f_1(x)f_2(x)$, it becomes

$$x_{n+1} = \beta [\alpha r_1 [\gamma r_1 x_n (1-x_n) + (1-\gamma)x_n] + (1-\beta)x_n, n \text{ is odd;}$$

$$x_{n+1} = \beta [r_2 [\alpha r_2 [\gamma r_2 x_n (1-x_n) + (1-\gamma)x_n] + (1-\alpha)x_n] + (1-\beta)x_n, n \text{ is even;}$$

where $x, r_1, r_2 \in \mathbb{R}$.

To underline the stable orbits of alternate logistic map in Noor orbit, we plot bifurcation diagrams at $\alpha = \beta = \gamma = 0.9$ and $\alpha = \beta = \gamma = 0.5$. See Figure. 4.4. From Figure.4.4, the range of convergence of the parameter r is increased in NO as α, β, γ tends to 0. Also, we observe that the chaotic range decreases as we decrease α, β, γ -value from Figure 4.4.

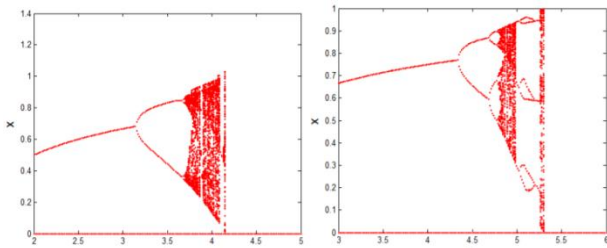


Fig. 4.4 Bifurcation diagram of the Noor logistic map at (a) $\alpha = \beta = \gamma = 0.9$ b) and $\alpha = \beta = \gamma = 0.5$.

They consider the logistic map in the alternated system defined in Eq. 3.2 to study the switching model in NO, i.e., even-odd switching strategy. Let

us fix a value of $r = r_0$, for the odd number of iterations. And they vary the value of r for even number of iterations and construct the bifurcation diagram. See a switched bifurcation diagram for the

logistic map in NO at ($\alpha = \beta = \gamma = 0.9, r_0 = 3.866$) in

Figure. 4.5. Here, the value of r_0 has been taken as chaotic value from Figure 3.4(a).

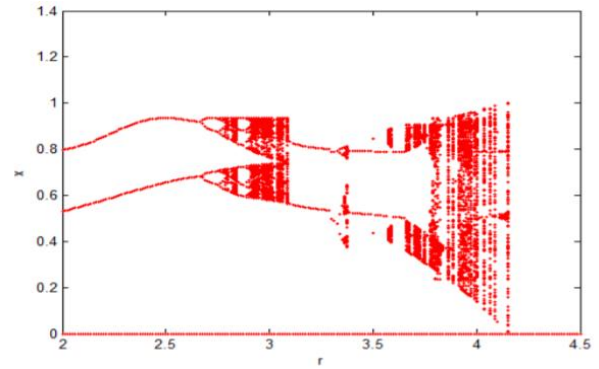


Fig. 4.5 Bifurcation diagram for the alternate Noor logistic map at ($\alpha = \beta = \gamma = 0.9, r_0 = 3.866$)

Also, they give examples of control of chaos by applying a switching strategy in the NO logistic map, where chaos + chaos results in order through

time series. They consider two chaotic values r_1 and r_2 for different $\alpha = \beta = \gamma$ -values to form “chaos1 + chaos2 = order” from above figures. At $\alpha = \beta = \gamma = 0.9, r_1 = 3.866, r_2 = 3.84$ forms 8-periodic orbit. See Fig.4.6. And at $\alpha = \beta = \gamma = 0.5, r_1 = 4.965, r_2 = 5.3$, we obtain 6-periodic orbit. See Fig. 4.7.

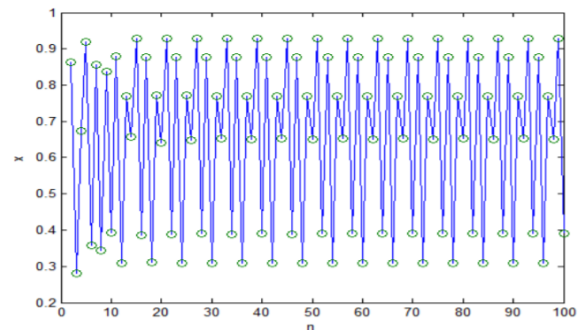


Fig. 4.6 8-periodic cycle time series at ($\alpha = \beta = \gamma = 0.9, r_1 = 3.866, r_2 = 3.84$).

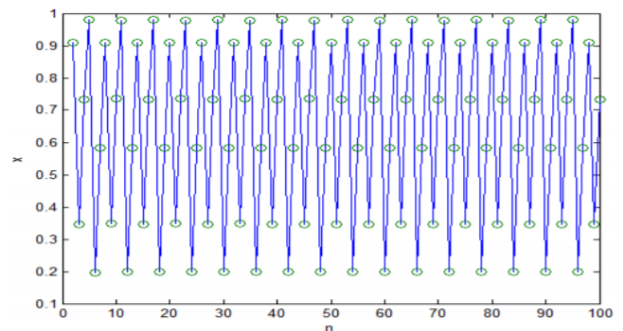


Fig. 4.7 6-periodic cycle time series at ($\alpha = \beta = \gamma = 0.5, r_1 = 4.965, r_2 = 5.3$)

In 2015 [14], Logistic map is studied in Noor orbit, and stability domain is increased, but still chaotic values are observed. So, by applying Parrondo's paradox to Noor logistic map the chaotic situation may be controlled.

5. TENT MAP

Consider the parametrized tent map which can be described by

$$T_r(x) = \begin{cases} 2rx, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2r(1-x), & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

where $0 < r \leq 2$. This map is continuous, linear in each of the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. Often tent map is introduced as one of the first example of chaotic maps literature for nonlinear discrete dynamical systems [2].

We will study T_1 which is the original tent map T, and which has some very interesting features.

The tent map $T : [0, 1] \rightarrow [0, 1]$ is given by

$$T(x) = \begin{cases} 2x, & \text{for } x \text{ in } [0, \frac{1}{2}] \\ 2-2x & \text{for } x \text{ in } [\frac{1}{2}, 1]. \end{cases}$$

The graph of T appears in figure 1. The major difference between the graph of T and the graph of T_r when $r < 1$ is the fact that the range of T fills out the whole interval [0,1]. The map T stretches the interval $[0, \frac{1}{2}]$ over the whole interval [0,1], and folds the interval $[\frac{1}{2}, 1]$ back to the whole interval [0,1].

Figure1 shows that the tent map T has 0 and $\frac{2}{3}$ two fixed points. Figure 2 show that $T^{[2]}$ has four fixed points.

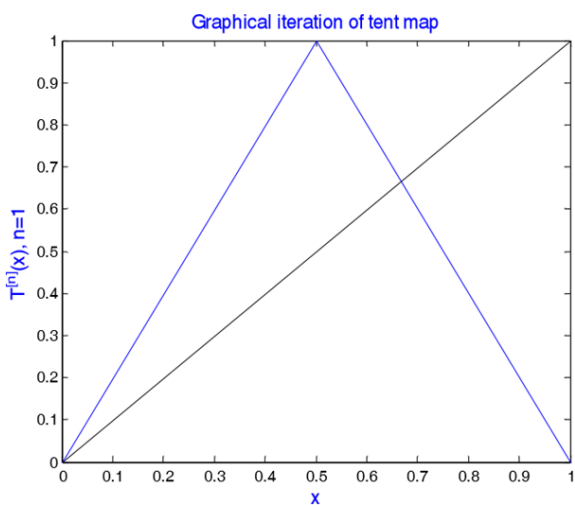


Fig. 1

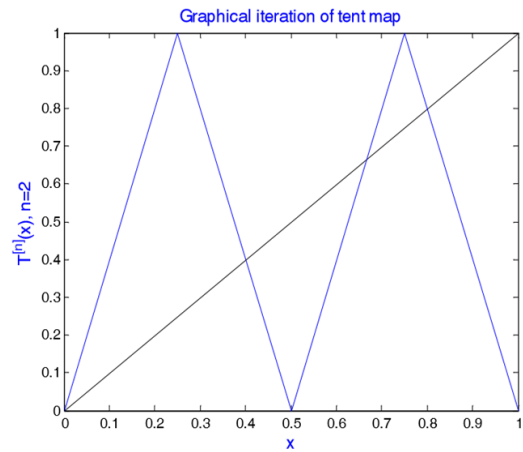


Fig. 2

After that, we choose an initial number x and calculate 200 iterations, $x = .230457$ and $r = 1$. Then the Figure 3 show the orbit of $x = .230457$.

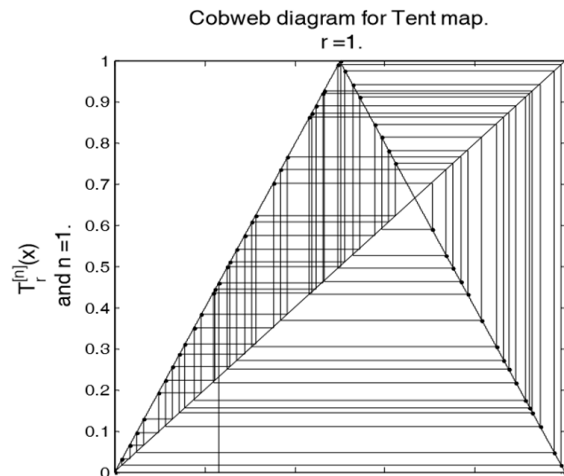


Fig. 3

This is a chaotic orbit for the tent map. This is a typical result for the typical initial condition when $r = 1$. We can land on one of the unstable periodic orbit, for some special choices of initial conditions.

Proposition. 1 [8]. The tent map T has sensitive dependence on initial conditions on $[0, 1]$.

Proposition. 2 [8]. The set of periodic points under T is dense in $[0, 1]$.

Proposition. 3 [8]. The tent map T is topological transitive on $[0, 1]$.

Therefore, by the definition of chaotic system given by Robert L. Devaney [3], tent map T has chaotic behavior on $[0, 1]$.

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