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An Analysis on Some New Fixed Point Theorem in Generalized Modular Metric Spaces

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Abstract – In this Paper, we first give a new fixed point theorem which is main theorem of our study in modular metric spaces. After that, by using this theorem, we express some interesting results. Moreover, we characterize completeness in modular metric spaces via this theorem. The aim of this paper is to prove the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings defined on generalized metric spaces with a graph. In this work, we discuss the definition of the Reich contraction single or multivalued mappings defined in a modular metric space. In our investigation, we prove the existence of fixed point results for these mappings. In this paper, we introduce a new concept of generalized modular metric space. Then we present some examples showing that the generalized modular metric space includes some kind of metric structures.

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INTRODUCTION

The existence and uniqueness of fixed point theorems of singlevalued maps have been a subject of great interest since Banach (1922) proved the well-known Banach contraction principle in 1922. This result is very interesting in its own right due to its applications like in computer science, physics, image processing engineering, economics, and telecommunication. Very early on, many mathematicians tried to find a multivalued version. Nadler (1950) was the one who successfully gave this extension. His result found many applications to differential inclusions, control theory, convex optimization, and economics. This is the reason why many authors have studied Nadlers fixed point result.

Reich's generalization of Nadler's fixed point result in S. Reich (1972) states that a mapping $T : X \to \mathcal{K}(X)$, where $\mathcal{K}(X)$ is the family of all nonempty compact subsets of *X*, has a fixed point if it satisfies $H(Tx, Ty) \leq k(d(x, y))d(x, y)$ for all $x, y \in X$ with $x \neq y$, where $k : (0, \infty) \to [0, 1)$ such that $\limsup_{r \to t+} k(r) < 1$ for every $t \in (0, \infty)$. In fact, Reich (1974) asked whether this result holds when *T* takes values in *CB(X)* instead of $\mathcal{K}(X)$, where $\mathcal{CB}(X)$ is the family of all nonempty closed and bounded subsets of *X.* In 1989, Mizoguchi and Takahashi pH] gave a partial answer to Reich's question.

Recently, Chistyakov (2010) has introduced the notion of modular metric spaces. This concept is a generalization of the classical modulars over linear spaces like Orlicz spaces. Moreover, the modular type conditions are natural and easily verified then their metric or norm equivalent. In A. A. N. Abdou, M. A. Khamsi (2013,2014), the authors initiated the fixed point theory in modular metric spaces. This work extends on these results where we discuss the definition of the Reich contraction single valued and multivalued mappings defined in modular metric spaces. In particular, we investigate the conditions under which such mappings have a fixed point.

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. In 1976, Jungck proved a common fixed point theorem for commuting maps such that one of them is continuous. In 1982, Sessa generalized the concept of commuting maps to weakly commutating pair of self-mappings. In 1986, Jungck generalized this idea, first to compatible mappings and then in 1996 to weakly compatible mappings. Using the weakly compatibility, several authors established coincidence points results for various classes of mappings on metric spaces with Fatou property. In 2011, Haghi et al. showed that some coincidence point and common fixed point generalizations in fixed point theory are not real generalizations as they could easily be obtained from the corresponding fixed point theorems.

In 1990, the fixed point theory in modular function spaces was initiated by Khamsi, Kozlowski, and Reich (1990).Modular function spaces are a special case of the theory of modular vector spaces introduced by Nakano (1950). Modular metric spaces were introduced in Chistyakov, V.V.(2010). Fixed point theory in modular metric spaces was studied by Abdou and Khamsi (2013). Their approach was fundamentally different from the one studied in Chistyakov, V.V.(2010). In this paper, we follow the same approach as the one used in Abdou, A.A.N., Khamsi, M.A.(2013).

Generalizations of standard metric spaces are interesting because they allow for some deep understanding of the classical results obtained in metric spaces. One has always to be careful when coming up with a new generalization. For example, if we relax the triangle inequality, some of the classical known facts in metric spaces may become impossible to obtain. This is the case with the generalized metric distance introduced by Jleli and Samet in Jleli, M., Samet, B.(2015). The authors showed that this generalization encompasses metric spaces, b-metric spaces, dislocated metric spaces, and modular vector spaces. In this paper, considering both a modular metric space and a generalized metric space in the sense of Jleli and Samet (2015), we introduce a new concept of generalized modular metric space.

Definition : Let X be an abstract set. A function $D:(0,\infty)\times X\times X\to[0,\infty]$ is said to be a regular generalized modular metric *(GMM)* on X if it satisfies the following three axioms:

(GMM_1) If $D_\lambda(x,y)=0$ for some $\lambda>0,$ then *x-y* for all $x,y\in X;$

(GMM_2) $D_\lambda(x,y)=D_\lambda(y,x)$ for all $\lambda>0$ and $x,y\in X;$

(GMM$_3$) There exists C > 0 such that, if $(x,y)\in X\times X, \{x_n\}\subset X$ with $\lim_{n\to\infty}D_\lambda(x_n,x)=0$ for some $\lambda>0,$ then $D_\lambda(x,y)\le C\limsup_{n\to\infty}D_\lambda(x_n,y).$

The pair *(X,D)* is said to be a generalized modular metric space *(GMMS).*

It is easy to check that if there exist *x,y*∈*X* such that there exists $\{x_n\}\subset X$ with $\lim_{n\to\infty}D_\lambda(x_n,x)=0$ for some $\lambda>0,$ and $D_\lambda(x,y)<\infty,$ then we must have $C\ge 1.$ In fact, throughout this work, we assume $C\ge 1.$ Let *D* be a *GMM* on *X.* Fix $x_0\in X.$ The sets

$$\begin{cases} X_D=X_D(x_0)=\{x\in X: D_\lambda(x,x_0)\to 0 \text{ as } \lambda\to\infty\} \\ X_D^*=\{x\in X:\exists\lambda=\lambda(x)>0 \text{ such that } D_\lambda(x,x_0)<\infty\}\end{cases}$$

are called *generalized modular sets.* Next, we give some examples that inspired our definition of a *GMMS.*

Example : (Modular vector spaces *(MVS)* [13]) Let A be a linear vector space over the field R. A function $\rho:X\to[0,\infty]$ is called regular modular if the following hold:

1. $\rho(x)=0$ if and only if *x*= 0,

2. $\rho(\alpha x)=\rho(x)$ if $|\alpha|=1,$

3. $\rho(\alpha x+(1-\alpha)y)\le\rho(x)+\rho(y)$ for any $\alpha\in[0,1],$

for any $x,y\in X$. Let ρ be regular modular defined on a vector space *X.* The set

$$X_\rho=\left\{x\in X; \lim_{\alpha\to 0}\rho(\alpha x)=0\right\}$$

is called a *MVS.* Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X_ρ and $x\in X_\rho$. If $\lim_{n\to\infty}\rho(x_n-x)=0,$ then $\{x_n\}_{n\in\mathbb{N}}$ is said to ρ-converge to *x.* ρ is said to satisfy the Δ_2-condition if there exists $K\neq 0$ such that $\rho(2x)\le K\rho(x)$ for any $x\in X_\rho$. Moreover, ρ is said to satisfy the Fatou property(FP) if $\rho(x-y)\le\liminf_{n\to\infty}\rho(x_n-y),$ whenever $\{x_n\}$ ρ-converges to *x* for any $x,y,x_n\in X_\rho$. Next, we show that a MVS may be embedded with a GMM structure. Indeed, let (X,ρ) be a *MVS.* Define $D:(0,+\infty)\times X\times X\to[0,+\infty]$ by

$$D_\lambda(x,y)=\rho\left(\frac{x-y}{\lambda}\right).$$

Then the following hold:

1. If $D_\lambda(x,y)=0$ for some $\lambda>0$ and any $x,y\in X,$ then *x=y,*

2. $D_\lambda(x,y)=D_\lambda(y,x)$ for any $\lambda>0$ and $x,y\in X;$

3. If ρ satisfies the FP, then for any $\lambda>0$ and {*„) such that $\{x_n/\lambda\}$ ρ-converges to $x/\lambda,$ we have

$$\rho\left(\frac{x-y}{\lambda}\right)\le\liminf_{n\to\infty}\rho\left(\frac{x_n-y}{\lambda}\right)\le\limsup_{n\to\infty}\rho\left(\frac{x_n-y}{\lambda}\right),$$

which implies

$$D_\lambda(x,y)\le\liminf_{n\to\infty}D_\lambda(x_n,y)\le\limsup_{n\to\infty}D_\lambda(x_n,y)$$

for any $x,y,x_n\in X_\rho$

Therefore, (*X,D*) satisfies all the properties of Definition as claimed. Note that the constant C which appears in the property *(GMMs)* is equal to 1 provided the FP is satisfied by ρ.

SOME FIXED POINTS THEOREMS IN MODULAR SPACES

In this section we discuss the existence of fixed points for mappings which are nonexpansive or contractive in the modular sense. Certainly, one can also consider mappings which are contractive with respect to the F-norm induced by the modular. It is worth to mention that, generally speaking, there is no natural relation between the two kinds of nonexpansiveness. Once again we would like to emphasize our philosophy that all the results expressed in terms of modulars are more convenient in the sense that their assumptions are much easier to verify.

Definition . Let *C* be a subset of a modular spaceand let $T : C \to C$ be an arbitrary mapping. L_ρ

1. *T* is a ρ-contraction if there exists $\lambda < 1$ such that $\rho(T(f) - T(g)) \leq \lambda\, \rho(f - g)$ for all $f, g \in C$.

2. *T* is said to be ρ-nonexpansive if $\rho(T(f) - T(g)) \leq \rho(f - g)$ for all $f, g \in C$

3. $f \in C$ is said to be a fixed point of *T* if *T(f) = f.* The fixed point set of *T* will be denoted *Fix(T).*

C will be said to have the fixed point property if every ρ-nonexpansive selfmap defined on *C* has a fixed point.

An analog to banach contraction principle, can be stated as follows.

Theorem . Let C be ρ-complete ρ-bounded subset of L_ρ and $T : C \to C$ be a ρ-strict contraction. Then T has a unique fixed point $z \in C$". Moreover z is the ρ-limit of the iterate of any point in C under the action of T.

Recall that a subset *D* of L_ρ is said to be p-complete if every ρ-Cauchy sequence from *D* is convergent in *D.*

We may relax the assumption regarding the boundedness of *C* and assume there exists a bounded orbit instead. In this case, the uniqueness of the fixed point is dropped and replaced by if *f* and *g* are two fixed points of *T* such that $\rho(f - g) < \infty$, then $f = g$

NEW FIXED POINT THEOREM IN MODULAR METRIC SPACES

The fixed point theory is used in many different fields of mathematics such as topology, analysis, nonlinear analysis and operator theory. Moreover, it can be applied to different disciplines such as statistics, economy, engineering, etc. In literature, studies of fixed point theory cover a wide range. The most basic and famous fixed point theorem is Banach fixed point theorem which was introduced in 1922. It guarantees the existence and uniqueness of solution of a functional equation. Besides Banach, many different fixed point theorems were introduced.

In 1950, Nakano introduced modular spaces. Then Chistyakov introduced the concept of modular metric spaces, which have a physical interpretation, via F-modulars in 2008 and he further developed the theory of these spaces in 2010. Then many authors made various studies on this structures.

In this paper, we first give a new fixed point theorem which is main theorem of our study. After that, by using this theorem, we express some interesting results. Moreover, we characterize completeness in modular metric spaces via this theorem. Finally, we use our main theorem to show the existence of solution for a specific problem in dynamic programming.

Here, we express a series of definitions of some basic concepts related to modular metric spaces.

Definition 1. Let X be a linear space on $\mathbb{R}$. If a functional $\rho : X \to [0, \infty]$ satisfies the following conditions, we call that ρ is a modular on X:

1. $\rho(0) = 0;$

2. If $x \in X$ and $\rho(\alpha x) = 0$ for all numbers $\alpha > 0$, then x = 0:

3. $\rho(-x) = \rho(x)$, for all $x \in X;$

4. $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ for all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$ and $x, y \in X.$

Let $X \neq \emptyset$ and $\lambda \in (0, \infty)$. Generally, a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is denoted as $\omega_\lambda(x, y) = \omega(\lambda, x, y)$ for all $x, y \in X$ and $\lambda > 0.$

Definition 2. Let $X \neq \emptyset.$ A function $\omega : (0, \infty) \times X \times X \to [0, \infty]$, which satisfies the following conditions for all $x, y, z \in X$, is called a metric modular on X:

(ml) $\omega_\lambda(x, y) = 0$ for all $\lambda > 0 \Leftrightarrow x = y;$

(m2) $\omega_\lambda(x, y) = \omega_\lambda(y, x)$ for all $\lambda > 0;$

(m3) $\omega_{\lambda+\mu}(x, y) \leq \omega_\lambda(x, z) + \omega_\mu(z, y)$ for all $\lambda, \mu > 0.$

If $0 < \mu < \lambda$, from properties of metric modular, we obtain that

$$\omega_\lambda(x, y) \leq \omega_{\lambda-\mu}(x, x) + \omega_\mu(x, y) = \omega_\mu(x, y)$$

for all $x, y \in X.$

From [10,11], we know that for a fixed $x_0 \in X$, the two sets

$$X_\omega = X_\omega(x_0) = \{x \in X : \omega_\lambda(x, x_0) \to 0 \text{ as } \lambda \to \infty\}$$

and

$X^*_\omega = X^*_\omega(x_0) = \{x \in X : \exists\lambda = \lambda(x) > 0$ such that $\omega_\lambda(x, x_0) < \infty\}$ are said to be modular spaces.

It is known that if ω is a metric modular on a nonempty set *X*, then the modular space X_ω can be equipped with a metric, generated by ω and given by $d_\omega(x, y) = \inf\{\lambda > 0 : \omega_\lambda(x, y) \le \lambda\}$ for all $x, y \in X_\omega$. The pair (X_ω, d_ω) is called a modular metric space.

FIXED POINT THEOREMS (FPT) IN GMMS

The following definition is useful to set new fixed point theory on GMMS.

Definition 1 Let (X_D, D) be a GMMS.

1. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be D-convergent to $x \in X_D$ if and only if $D_\lambda(x_n, x) \to 0,$ as $n \to \infty,$ for some $\lambda > 0.$

2. The sequence $\{x_n\}_{n\in\mathbb{N}}$ in X_D is said to be *D-Cauchy* if $D_\lambda(x_m, x_n) \to 0,$ as $m, n \to \infty,$ for some $\lambda > 0.$

3. A subset C of X_D is said to be D-closed if for any $\{x_n\}$ from C which D-converges to $x, x \in C.$

4. A subset C of X_D is said to be D-complete if for any x_n) D-Cauchy sequence in C such that $\lim_{n,m\to\infty} D_\lambda(x_n, x_m) = 0$ for some λ, there exists a point $x \in C$ such that $\lim_{n,m\to\infty} D_\lambda(x_n, x) = 0.$

5. A subset C of X_D is said to be D-bounded if, for some $\lambda > 0,$ we have $\delta_{D,\lambda}(C) = \sup\{D_\lambda(x, y); x, y \in C\} < \infty.$

In general, if $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for some $\lambda > 0,$ then we may not have $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for all $\lambda > 0.$ Therefore, as it is done in modular function spaces, we will say that *D* satisfies Δ_2-condition if and only if $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for some $\lambda > 0$ implies $\lim_{n\to\infty} D_\lambda(x_n, x) = 0$ for all $\lambda > 0.$

Another question that comes into this setting is the concept of D-limit and its uniqueness.

Proposition 2.1 *Let {Xd, D) be a* GMMS. *Let [x*$_n$) *be a sequence in Xd. Let* $(x, y) \in X_D \times X_D$ *such that* $D_\lambda(x_n, x) \to 0$ *and* $D_\lambda(x_n, y) \to 0$ *as* $n \to \infty$ *for some* $\lambda > 0.$ *Then x-y.*

REICH TYPE MAPPINGS IN MODULAR METRIC SPACES

Definition 1. Let (X, ω) be a modular metric space and *M* be a nonempty subset of X_ω. The map $T : M \to M$ is called a Reich contraction if there exists $k : (0, +\infty) \to [0, 1)$ which satisfies $\limsup_{s\to t^+} k(s) < 1$ for any $t \in [0, +\infty),$ such that for any distinct elements $a, b \in M,$ we have

$$\omega_1(T(a), T(b)) \le k(\omega_1(a, b))\ \omega_1(a, b).$$

A point *a* is said to be a fixed point of *T* if $T(a) = a.$

Theorem 1. Let (X, ω) be a modular metric space where ω is a convex regular modular. Assume that ω satisfies the Δ_2-type condition. Let C be an ω-complete nonempty subset of X_ω. Let $T : C \to C$ be a Reich contraction mapping. Then, T has a unique fixed point $x \in C$ and $\{T^n(z)\}$ ω-converges to x for any $z \in C.$

Proof. The definition of Reich contraction implies the existence of $k : (0, +\infty) \to [0, 1)$ which satisfies $\limsup_{s\to t^+} k(s) < 1$ for any $t \in [0, +\infty),$ such that for any different $x, y \in C$ $\omega_1(T(x), T(y)) \le k(\omega_1(x, y))\ \omega_1(x, y).$

It is clear that *T* has at most one fixed point since ω is regular. Next we investigate the existence of a fixed point. Fix $x_0 \in X.$ If $T^n(x_0)$ is a fixed point of *T* for some $n \in \mathbb{N},$ then we have nothing to prove. Otherwise assume that $T^{n+1}(x_0) \ne T^n(x_0)$ for any $n \in \mathbb{N}.$ Since

$$\omega_1(T^{n+1}(x_0), T^n(x_0)) \le k(\omega_1(T^n(x_0), T^{n-1}(x_0)))\ \omega_1(T^n(x_0), T^{n-1}(x_0)),$$

We conclude that $\omega_1(T^{n+1}(x_0), T^n(x_0)) < \omega_1(T^n(x_0), T^{n-1}(x_0))$ for any $n \in \mathbb{N}.$ Hence the sequence of positive numbers $\{\omega_1(T^{n+1}(x_0), T^n(x_0))\}$ is convergent. Set $t_0 = \lim_{n\to+\infty} \omega_1(T^{n+1}(x_0), T^n(x_0)) = \inf_{n\in\mathbb{N}} \omega_1(T^{n+1}(x_0), T^n(x_0)).$

Since $\limsup_{s\to t_0^+} k(s) < 1,$ there exist $\alpha < 1$ and $n_0 \ge 1$ such that $k(\omega_1(T^{n+1}(x_0), T^n(x_0))) \le \alpha$ for any $n \ge n_0.$ Then, we have

$$\omega_1(T^{n+1}(x_0), T^n(x_0)) \le \prod_{k=n_0}^{k=n} k(\omega_1(T^{k+1}(x_0), T^k(x_0)))\ \omega_1(T^{n_0+1}(x_0), T^{n_0}(x_0))$$
$$\le \alpha^{n-n_0}\ \omega_1(T^{n_0+1}(x_0), T^{n_0}(x_0))$$

for any $n \ge n_0$. Lemma 2.6 implies that $\{T^n(x_0)\}$ is ω-Cauchy. Using the ω-completeness of *C,* we conclude that $\{T^n(x_0)\}$ ω-converges to some $x \in C.$

Next we show that a: is a fixed point of *T*. Note that we have

$$\begin{aligned}\omega_2(x,T(x)) &\le \omega_1(x,T^n(x_0)) + \omega_1(T^n(x_0),T(x)) \\ &\le \omega_1(x,T^n(x_0)) + k(\omega_1(T^{n-1}(x_0),x))\omega_1(T^{n-1}(x_0),x) \\ &\le \omega_1(x,T^n(x_0)) + \omega_1(T^{n-1}(x_0),x)\end{aligned}$$

For any $n \ge 1$. Since $\{T^n(x_0)\}$ ω-converges to *x*, we deduce that $\omega_2(x,T(x)) = 0$. The regularity of ω implies that $T(x) = x$. The uniqueness of the fixed point of *T* will imply that $\{T^n(z)\}$ ω-converges to *x* for any $z \in C$.

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