Journal of Advances and Scholarly Researches in Allied Education
Vol. 16, Issue No. 1, January-2019, ISSN 2230-7540

An Analysis on Some Basic Boundary Value Problems: A Case Study of Complex Partial Differential Equations

Versha Chopra*

Abstract – In this paper, the ongoing outcomes on basic boundary value problems of complex analysis are studied for complex model equations of subjective order on just connected bounded domains, especially in the unit plate, on unbounded domains, for example, upper half plane and upper right quarter plane and on duplicate connected domains containing circular rings.

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INTRODUCTION

The examinations on the boundary value problems for complex differential equations had another beginning stage for complex differential equations in the nineteenth century. Riemann has expressed the issue "Discover a function $w(z) = u + iv$ analytic in the space Ω, which satisfies at each boundary point the relation

$$F(u, v) = 0 \ \ (on\ \partial\Omega)\text{"}$$

Afterward, this announcement is known as Riemann issue. Be that as it may, he expressed just some broad contemplations in regards to with the solvability of the problem. The speculation of Riemann issue to a linear first-order differential equation together with the linear boundary condition

$$\alpha u + \beta v = \text{Re}[\overline{\lambda(z)}w] = \gamma \text{ on } \partial\Omega$$

was considered by Hilbert (1924). This new type of the issue is called as the generalized Riemann-Hilbert issue. A few augmentations of this issue has been treated by numerous specialists.

The first-order linear complex partial differential equation

$$w_{\bar{z}} = a(z)w(z) + b(z)\overline{w(z)}$$

has been considered by Yekua (1962) independently and all the while. Its answers are called as summed up analytic (or pseudo-holomorphic) functions. The boundary value problems depicted above and their specific cases known as Schwarz. Dirichlet. Neumann and Robin problems are treated by numerous scientists.

Boundary value problems for higher-order linear complex partial differential equations picked up fascination over the most recent twelve years. Dirichlet. Neumann, Robin. Schwarz and blended boundary value problems for model equations, that is for the equations of the structure $\partial_z^m \partial_{\bar{z}}^n w = f(z)$, are presented in the unit circle of the complex plane by Begehr (2005). In this article we need to give a coordinated study of the pertinent writing on the boundary value problems of complex analysis, and uncover a few problems which are as yet open.

DIRICHLET PROBLEM FOR COMPLEX PDES

Just Connected Bounded Domain Case-

Numerous creators have researched the Dirichlet issue in just connected domains. To give the unequivocal portrayals for the arrangements of the problems, we will consider the specific case of the unit plate $\mathbb{D}$ of the complex plane. Give us a chance to begin by giving the related consonant and polyhannonic Green functions.

In $\mathbb{D}$, the consonant Green function is characterized as

$$G_1(z, \zeta) = \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2$$

what's more, its properties are given in H. Begehr (2006 – 08). A polyhannonic Green function G_n is characterized iteratively by

$$G_n(z,\zeta) = -\frac{1}{\pi}\iint_{\mathbb{D}} G_1(z,\tilde{\zeta})G_{n-1}(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

for $n \geq 2$. The explicit expressions of $G_n(z,\zeta)$ for n=2 and for n = 3 are given in H. Begehr (2006 – 08). separately. $G_n(z,\zeta)$ are utilized to take care of the accompanying n-Dirichlet problems for the n-Poisson equation.

Theorem 1. The Dirichlet issue

$$(\partial_z\partial_{\bar{z}})^n w = f \text{ in } \mathbb{D},\ (\partial_z\partial_{\bar{z}})^\mu w = \gamma_\mu,\ 0 \leq \mu \leq n-1 \text{ on } \partial\mathbb{D}$$

$f \in L^1(\mathbb{D}), \gamma_\mu \in C(\partial\mathbb{D}),\quad 0 \leq \mu \leq n-1$ is remarkably reasonable. The arrangement is

$$w(z) = -\sum_{\mu=1}^{n}\frac{1}{4\pi i}\int_{\partial\mathbb{D}} \partial_{\nu_\zeta}G_\mu(z,\zeta)\gamma_{\mu-1}(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\iint_{\mathbb{D}} G_n(z,\zeta)f(\zeta)d\xi d\eta. \quad (1)$$

The unequivocal types of the arrangements (1) are given in H. Begehr (2006 – 08) on account of n = 2. In A. Kumar and R. Prakash (2012). creators considered the issue given in Theorem 2.1 and they gave the express portrayal of the exceptional arrangement utilizing the iterative wholes for n = 2 and n = 3 are given. Begehr. Du and Wang (2008) tackled the Dirichlet issue for polyhannonic functions by utilizing the deterioration of polyhannonic functions and changing the issue into an identical Riemann boundary value issue for polyanalytic functions. In H. Begehr and Y.Wang, (2007), creators tackled the Dirichlet issue researched in Begehr. Du and Wang (2008) by another methodology. The unequivocal articulation of the one of a kind answer for the Dirichlet issue of triharmonic functions in the unit circle is gotten by utilizing the alleged frail decay of polyhannonic functions and changing over the issue into an identical Dirichlet boundary value issue for analytic functions. Rather than the boundary condition as indicated by Begehr. Du and Wang (2008). the necessity of smoothness for the given functions is decreased.

Basically Connected Unbounded Domain Case-

In this subsection we give an outline of the problems characterized in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : 0 < \text{Im}\, z\}$ and in right upper quarter plane $\text{Re}\ z, 0 < \text{Im}\ z\}$.

The polyhannonic Green function for $\mathbb{H}$ is given utilizing the Almansi extension:

$$\tilde{G}_n(z,\zeta) = \frac{|\zeta - z|^{2(n-1)}}{(n-1)!^2}\log\left|\frac{\bar{\zeta}-z}{\zeta - z}\right|^2 - \sum_{\mu=1}^{n-1}\frac{1}{\mu(n-1)!^2}|\zeta - z|^{2(n-1-\mu)}(\zeta-\bar{\zeta})^\mu(z-\bar{z})^\mu.$$

Begelir and Gaertner (2007) have demonstrated the accompanying theorem.

Theorem 1. For given f fulfilling $|z|^{2(n-1)}f(z) \in L_1(\mathbb{H};\mathbb{C})$, $\gamma_\nu \in C^{n-2\nu}(\mathbb{R};\mathbb{C})$ for $0 \leq 2\nu \leq n-1$, $\hat{\gamma}_\nu \in C^{n-1-2\nu}(\mathbb{R};\mathbb{C})$ for $0 \leq 2\nu \leq n-2$ with the separate subsidiaries bounded, the Dirichlet issue

$$(\partial_{\bar{z}}\partial_z)^n w = f \ \ in\ \ \mathbb{H}$$

$$(\partial_z\partial_{\bar{z}})^\nu w = \gamma_\nu \ \ for\ \ 0 \leq 2\nu \leq n-1,$$

$$\partial_z^\nu\partial_{\bar{z}}^{\nu+1} w = \hat{\gamma}_\nu \ \ for\ \ 0 \leq 2\nu \leq n-2 \ on\ \mathbb{R}$$

Is particularly resolvable in a frail sense by

$$\begin{aligned} w(z) = \ & \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Bigg[\left(\frac{z-\bar{z}}{t-\bar{z}}\right)^n\frac{\gamma_0(t)}{t-z} + \sum_{\mu=1}^{n-1}(-1)^\mu\frac{(z-\bar{z})^\mu}{\mu}g_\mu(z,t)\hat{\gamma}_0(t) \\ & + \sum_{\nu=1}^{[\frac{n-1}{2}]}\Bigg[\sum_{\mu=2\nu}^{n-1}(-1)^{\mu-\nu}\frac{(\mu-\nu-1)!}{\mu!(\nu-1)!}\frac{(z-\bar{z})^\mu}{(t-z)^{\mu-2\nu+1}} \\ & + \sum_{\mu=2\nu}^{n-1}(-1)^{\nu-1}\frac{(\mu-\nu)!}{\mu!\nu!}\frac{(z-\bar{z})^\mu}{(t-\bar{z})^{\mu-2\nu+1}}\Bigg]\gamma_\nu(t) \\ & + \sum_{\nu=1}^{[\frac{n}{2}]-1}\sum_{\mu=2\nu+1}^{n-1}(-1)^{\mu-\nu}\frac{(\mu-\nu-1)!}{\mu!\nu!}(z-\bar{z})^\mu g_{\mu-2\nu}(z,t)\hat{\gamma}_\nu(t)\Bigg]dt \\ & - \frac{1}{\pi}\iint_{\mathbb{H}} G_n(z,\zeta)f(\zeta)d\xi d\eta, \end{aligned}$$

where for $1 \leq \alpha$,

$$g_\alpha(z,\zeta) = \frac{1}{(\bar{\zeta}-z)^\alpha} + \frac{(-1)^\alpha}{(\zeta - \bar{z})^\alpha}.$$

In $\mathbb{Q}_1 = \{z \in \mathbb{C} : 0 < \text{Re}\ z, 0 < \text{Im}\ z\}$, the following result for the Dirichlet boundary value issue is given for the inhomogeneous Cauchy-Riemann equation in H. Begehr and G. Harutjunjan (2006).

NEUMANN PROBLEM FOR COMPLEX PDES

Simply Connected Bounded Domain Case-

The consonant Neumann function for the area $\mathbb{D}$ is given by

$$N_1(z,\zeta) = \log|(\zeta - z)(1 - z\bar{\zeta})|^2 \quad (1)$$

for $z, \zeta \in \mathbb{D}$. (1) fulfills

$$\partial_{\nu_z}N_1(z,\zeta) = (z\partial_z + \bar{z}\partial_{\bar{z}})N_1(z,\zeta) = 2 \quad (2)$$

for $z \in \partial\mathbb{D}, \zeta \in \mathbb{D}$. But the higher-order Neumann functions are difficult to discover in their unequivocal structures. They might be characterized iteratively for $n \in \mathbb{N}$ where $n \geq 2$, as

$$N_n(z,\zeta) = \frac{1}{\pi}\iint_{\mathbb{D}} N_1(z,\tilde{\zeta})N_{n-1}(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}. \quad (3)$$

For the unequivocal structure on account of n = 2 and n = 3. By the guide of (3), the higher-order

Poisson equation is explored under the Neumann conditions and the accompanying outcome is gotten.

Theorem 1. The Neumann-n issue

$$(\partial_z\partial_{\bar z})^n w = f \text{ in } \mathbb{D}\,,\ f\in L^p(\mathbb{D}) \text{ for } 1<p<+\infty,$$
$$\partial_\nu(\partial_z\partial_{\bar z})^\sigma w = \gamma_\sigma \text{ on } \partial\mathbb{D}\,,\ \gamma_\sigma\in C(\partial\mathbb{D}) \text{ for } 0\le\sigma\le n-1,$$

fulfilling

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(\partial_\zeta\partial_{\bar\zeta})^\sigma w(\zeta)\frac{d\zeta}{\zeta}=c_\sigma\,,\ c_\sigma\in\mathbb{C} \text{ for } 0\le\sigma\le n-1,$$

is feasible if and just if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_\sigma(\zeta)\frac{d\zeta}{\zeta}=\sum_{\mu=\sigma+1}^{n-1}\alpha_{\mu-\sigma}c_\mu+\frac{1}{\pi}\iint_{\mathbb{D}}\partial_{\nu_z}N_{n-\sigma}(z,\zeta)f(\zeta)d\xi d\eta\,. \quad (4)$$

Here $\alpha_1=2$ and for $3\le k$

$$\alpha_{k-1}=-\sum_{\mu=[\frac{k}{2}]}^{k-2}\frac{\mu!^2}{(k-1)!(k-1-\mu)!^2(2\mu-k+1)!}\alpha_\mu. \quad (5)$$

The arrangement is extraordinary and given by

$$w(z)=\sum_{\mu=0}^{n-1}\left\{\frac{1}{2}c_\mu\partial_{\nu_\zeta}N_{\mu+1}(z,\zeta)-\frac{1}{4\pi i}\int_{\partial\mathbb{D}}N_{\mu+1}(z,\zeta)\gamma_\mu(\zeta)\frac{d\zeta}{\zeta}\right\}$$
$$+\frac{1}{\pi}\iint_{\mathbb{D}}N_n(z,\zeta)f(\zeta)d\xi d\eta\,.$$

Especially, for the inhomogeneous biharmonic equation, closely resembling outcomes are pre¬sented in H. Begehr (2006 – 08) . We may counsel with H. Begehr (2004) for the arrangements of Bitsadze equation under Neumann conditions.

The inhomogeneous polyanalytic equation with the half-Neumann conditions $z\partial_{\bar z}^\nu\partial_z w=\gamma_\nu$ on $\partial\mathbb{D}, \partial_{\bar z}^\nu w(0)=c_\nu$ is remarkably tackled with some solvability conditions in H. Begehr and A. Kumar (2005).

Unbounded Domain Case-

The Neumann boundary value issue is considered for the inhomogeneous C'auchy-Riemann equation in a quarter plane and the solvability conditions and arrangements are given in unequivocal structure in H. Begehr and G. Harutjunjan (2006).

Neumann Problem. Let $f\in L_{p,2}(\mathbb{Q}_1;\mathbb{C})\cap C^\alpha(\overline{\mathbb{Q}}_1;\mathbb{C})$ for $2<p, 0<\alpha<1, \gamma_1,\gamma_2\in C(\mathbb{R}^+;\mathbb{C})$ with the end goal that $(1+t)^\delta\gamma_1(t), (1+t)^\delta\gamma_2(t),$ $(1+t)^\delta f(t), (1+t)^\delta f(it)$ are bounded for about $0<\delta, c\in\mathbb{C}$. Find $w\in C^1(\overline{\mathbb{Q}}_1;\mathbb{C})$ fulfilling $w_{\bar z}=f$ in $\mathbb{Q}_1, \partial_y w=\gamma_1$ for $0<x, y=0,\ \partial_x w=\gamma_2$ for $0<y, x=0, w(0)=c.$

Theorem 1. The Neumann issue is interestingly resolvable in the feeble sense if and if for any $z\notin\overline{\mathbb{Q}}_1$

$$\frac{1}{2\pi}\int_0^{+\infty}[\gamma_1(t)+if(t)]\frac{dt}{t-z}+\frac{1}{2\pi i}\int_0^{+\infty}[\gamma_2(t)-f(it)]\frac{dt}{t+iz}+\frac{1}{\pi}\iint_{\mathbb{Q}_1}f(\zeta)\frac{d\xi d\eta}{(\zeta-z)^2}=0 \quad (6)$$

holds. The arrangement is

$$w(z)=c+\frac{1}{2\pi}\int_0^{+\infty}[\gamma_1(t)+if(t)]\log\left|\frac{t^2-z^2}{t^2}\right|^2dt$$
$$+\frac{1}{2\pi}\int_0^{+\infty}[\gamma_2(t)-f(it)]\log\left|\frac{t^2+z^2}{t^2}\right|^2dt-\frac{z}{\pi}\iint_{\mathbb{Q}_1}\frac{f(\zeta)}{\zeta}\frac{d\xi d\eta}{\zeta-z}. \quad (7)$$

Likewise, in the upper half plane the Neumann issue is considered for the inhomoge¬neous Cauchy-Riemaim equation and Poisson equation.

SCHWARZ PROBLEM FOR COMPLEX PDES

Basically Connected Domain Case-

The first article in Schwarz issue for analytic functions. The fol¬lowing theorem gives the one of a kind arrangement of the Schwarz issue for inhomogeneous polvanalytic equation.

Theorem 1. The Schwarz issue for the homogeneous polyanalytic equation in the unit plate $\mathbb{D}$ characterized by

$\partial_{\bar z}^k w=f$ in $\mathbb{D}$, $\mathrm{Re}\ \partial_{\bar z}^l w=0$ on $\partial\mathbb{D}$, Im $\partial_{\bar z}^l w(0)=0\,,\ 0\le l\le n-1\,,$

is exceptionally resolvable for $f\in L^1(\mathbb{D})$. The arrangement is

$$w(z)=\frac{(-1)^k}{2\pi(k-1)!}\iint_{\mathbb{D}}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z}+\frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right)(\zeta-z+\overline{\zeta-z})^{k-1}d\xi d\eta\,.$$

Already the cases of k = 1 and k = 2 have been examined.

Unbounded Domain Case-

In the upper right quarter plane, the accompanying issue is characterized and explained by Abdy-manapov et al (2005).

Schwarz Problem. Let $f\in L_1(\mathbb{Q}_1;\mathbb{C}), \gamma_1,\gamma_2\in C(\mathbb{R}^+;\mathbb{R})$ be bounded on $\mathbb{R}^+=(0,+\infty)$. Find an answer of $w_{\bar z}=f$ in $\mathbb{Q}_1$ fulfilling

Re $w=\gamma_1$ on $0<x, y=0,$

Im $w=\gamma_2$ on $0<y, x=0.$

Theorem 1. The Schwarz issue is exceptionally feebly resolvable. The arrangement is

$$w(z) = \frac{2}{\pi i}\int_0^{+\infty} \gamma_1(t)\frac{z}{t^2-z^2}dt - \frac{2}{\pi i}\int_0^{+\infty} \gamma_2(t)\frac{z}{t^2+z^2}dt - \frac{2}{\pi}\iint_{Q_1}\left[\frac{zf(\zeta)}{\zeta^2-z^2} - \frac{z\overline{f(\zeta)}}{\bar{\zeta}^2-z^2}\right]d\xi d\eta. \quad (1)$$

On account of upper half plane the accompanying outcome is gotten in E. Gaertner (2006).

REFERENCES

1. A. Kumar and R. Prakash (2012). Dirichlet problem for inhomogeneous polyharmonic equation, *Complex Var. Elliptic Eqs.*, 53, pp. 643–651.

2. D. Hilbert (1924). *Grundz¨uge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig-Berlin.

3. E. Gaertner (2006). Basic Complex Boundary Value Problems in the Upper Half Plane, PhD thesis, FU Berlin.

4. H. Begehr and A. Kumar (2005). Boundary value problems for the inhomogeneous polyanalytic equation, *Analysis*, 25, pp. 55–71.

5. H. Begehr and E. Gaertner (2007). Dirichlet problem for the inhomogeneous polyharmonic equation in the upper half plane, *Georgian Math. J.*, 14, pp. 33–52.

6. H. Begehr and G. Harutjunjan (2006). Complex boundary value problems in a quarter plane, *Complex Analysis and Applications*, Proc. 13th Intern. Conf. Finite or Infinite Complex Analysis and Appl., Shantou, China, 2005, Y. Wang, S. Wu, H. Wulan, L. Yang (Eds.), World Sci., New Jersey, pp. 1–10.

7. H. Begehr and Y. Wang (2007). A new approach for solving a Dirichlet problem for polyharmonic functions, *Complex Var. Elliptic Eqs.*, 52, pp. 907–920.

8. H. Begehr (2007). A particular polyharmonic Dirichlet problem, *Complex Analysis Potential Theory*, T. Aliyev Azesogla, P. M. Tamrazov (Eds.), World Sci., Singapore, pp. 84–115.

9. H. Begehr (2006). Biharmonic Green functions, *Le Matematiche*, LXI, pp. 395–405.

10. H. Begehr (2004). Boundary value problems for the Bitsadze equation, *Memoirs Diff. Eqs. Math. Phys.*, 33, pp. 5–23.

11. H. Begehr, J. Du, and Y. Wang (2008). A Dirichlet problem for polyharmonic functions, *Ann. Math. Pura Appl.*, 187, pp. 435–457.

12. H. Begehr (2008). Six biharmonic Dirichlet problems in complex analysis, in: Function Spaces in Complex and Clifford Analysis, Natl. Univ. Publ. Hanoi, Hanoi, pp. 243–252.

13. H. Begehr (2005). Boundary value problems in complex analysis I, II, *Boletin de la Asosiaci´on Matematica Venezolana*, XII, pp. 65–85, pp. 223–241.

14. H. Begehr, T. Vaitekhovich (2009). Harmonic boundary value problems in half disc and half ring, Functiones et Approximatio, 40.2, pp. 251-282.

14. H. Begehr, T. Vaitekhovich (2011). How to find harmonic Green function in the plane, Comp.Var.Ellip.Eq., Vol.56(12), pp. 1169-1181.

15. I. N. Vekua (1962). *Generalized Analytic Functions*, Pergamon Press.

16. M. S. Akel, H. S. Hussein (2012). Two basic boundary value problems for inhomogeneous Cauchy-Riemann equation in an in_nite sector, Adv.Pure and Appl.Math., 3, pp. 315-328.

17. S. A. Abdymanapov, H. Begehr, A. B. Tungatarov (2005). Some Schwarz problems in a quarter plane, *Eurasian Math. J.*, No 3,pp. 22–35.

18. S. Burgumbayeva (2009). Boundary value problems for tri-harmonic functions in the unit disc, Ph.D. thesis, FU Berlin.

19. Y. Wang (2011). Boundary Value Problems for Complex Partial Differential Equations in Fan-Shaped domains, PhD thesis, FU Berlin.

Corresponding Author

Versha Chopra*