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Different Phase Study of Quantum Field Theory

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ABSTRACT

Series expansions play a major role in helping us extract results from many models which are too complex to solve directly. Traditional perturbation theory, for instance, can be successfully applied to Quantum Electrodynamics. Yet, it completely fails to provide convergent expansions when applied to theories where coupling constants are (in some sense) large. Hence the importance of non-perturbative series expansions, of which the Linear Delta Expansion (LDE) is one. In this paper, we study the application of the LDE to two very different models: the lattice scalar self-interacting field theory, and the dynamics of a quantum mechanical inflationary model. We will also develop sophisticated arbitrary precision numerical methods to aid us in pushing the expansion to reasonably high orders.After presenting an overview of the LDE, we shall apply it to the lattice theory. In particular, we will focus on the critical behaviour of the model, which will include the calculation of various critical exponents. We shall find that the LDE gives good qualitative results and clearly identifies the symmetry breaking aspects of the theory. On the other

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hand, we shall find that the quantitative results, especially those of critical behaviour, reproduce the results obtained by the much less sophisticated mean field theory.

The second model studied in the paper is the quantum dynamics of an inflationary model, often called the quantum mechanical slow-roll. A recent LDE study of the same model successfully tracks the system while in the inflation phase, but fails to follow suit into the reheating phase. Our aim will be to improve the methodology (and consequently the results) of that study, by employing a physically more intuitive criterion for optimizing the parameters of the theory. We will find, however, that the hoped for improvements remain elusive.

Introduction

The Method of Sources

We start by introducing the all important *partition function Z*, the quantity which contains all physical information about the system it describes:

$$Z = \mathrm{tr}\, e^{-S} \qquad (1.1)$$

The trace is performed over all degrees of freedom of the system, and the quantity denoted by S is the *action*, which describes the dynamics of the system. From the perspective of *statistical mechanics* we are used to seeing βH in place of the action.

β is the inverse temperature and H is the Hamiltonian of the system. Our problem, however, is one of *quantum field theory*, where temperature is irrelevant and actions are used to describe dynamics.

Since we study a lattice model in this thesis, we shall gear this introduction towards discrete actions. Thus we let the action depend on a set of fields $\{\phi_i\}$, where $i \in \Lambda$, and A is the set of all lattice sites.

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We denote the total number of lattice sites by $N - |\Lambda|$. In the well known magnetic models, ϕ_i is usually restricted to a discrete set of values, e.g. the Ising model allows $\phi_i - \pm 1$. For us, the fields ϕ_i will be unrestricted and continuous. Apart from those, the action will also depend on some set of coupling constants, which are of no importance to us at the moment.

We shall now add an explicit linear term to the action in equation (1.1), coupled to an inhomogeneous source J. In practice, inhomogeneous simply means that we have a set of sources $\{J_i\}$, one for each lattice site i. In standard magnetic theories,

for example, it is the external magnetic field which takes on the role of the source. With this added term, we have a slightly modified partition function:

$$Z - \mathrm{tr}\, e^{-S + \sum_{i \in \Lambda} J_i \phi_i} \qquad (1.2)$$

A partition function written as above is also called a *generating function*, due to the ease with which we can extract physical information about the system. The key is to differentiate the partition function with respect to the source to generate physical quantities. Even if there is no physical source present in the system, it is still used for algebraic convenience and then set to zero at the end of the calculation. Hence the name *method of sources* [1].

$$\langle \phi_p \rangle - \frac{1}{Z} \mathrm{tr} \left[\phi_p e^{-S + \sum_{i \subset \Lambda} J_i \phi_i} \right] \qquad (1.3)$$

Using the generating function, $\langle \phi_p \rangle$ can be neatly written as

$$\langle \phi_p \rangle \quad \frac{1}{Z} \frac{\partial Z}{\partial J_p} \quad \frac{\partial (\ln Z)}{\partial J_p} \qquad (1.4)$$

This motivates the definition of the *free energy F*

in terms of which the expectation value becomes simply

$$\langle \phi_p \rangle = -\frac{\partial F}{\partial J_p} \qquad (1.6)$$

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The response of the expectation value at the site *p* with respect to a variation of the source at site *q* is given by

$$\frac{\partial\langle\phi_p\rangle}{\partial J_q} - -\frac{\partial^2 F}{\partial J_p \partial J_q} - \langle\phi_p\phi_q\rangle - \langle\phi_p\rangle\langle\phi_q\rangle \qquad (1.7)$$

The expectation value of the field ϕ at a certain lattice site *p* is given by

This quantity is often called the *correlation function* of the fields ϕ_p and ϕ_q, and is denoted by G_{pq}. We can also write it in a physically more transparent form:

$$G_{pq} - \langle\phi_p\phi_q\rangle - \langle\phi_p\rangle\langle\phi_q\rangle - \langle\left(\phi_p - \langle\phi_p\rangle\right)\left(\phi_q - \langle\phi_q\rangle\right)\rangle \qquad (1.8)$$

which emphasizes that G_{pq} is a measure of the fluctuations of the field ϕ_p in response to fluctuations of the field ϕ_q. A related quantity is the *susceptibility*, denoted by χ, which is the sum of the correlations over the whole lattice:

$$\chi \quad \frac{1}{N}\sum_{i\in\Lambda}\sum_{j\in\Lambda} G_{ij} \qquad (1.9)$$

Note that we have defined the susceptibility χ as an intensive quantity by dividing with the volume of the system — in this case the number of lattice sites. Broadly speaking, a quantity is *intensive* if it is independent of the volume of the system, i.e. it is a kind of 'density' measure. Conversely, an *extensive* quantity depends on the volume, i.e. it is a kind of 'how much' measure.

In magnetic theories it is usual to define the *magnetization*, which amounts to the average of the spin expectation values over the whole lattice. By analogy, we define the average field, denoted by Φ:

$$\Phi = \frac{1}{N}\sum_{i\in\Lambda}\langle\phi_i\rangle \qquad (1.10)$$

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In practice, we usually work with translationally invariant theories which have a homogeneous source. To explore these conditions we have to set Jj = J for all $i \in \Lambda$ and the linear term becomes $\sum_{i\in\Lambda} J_i\phi_i \longrightarrow J\sum_{i\in\Lambda}\phi_i$. Due to symmetry considerations

(of translational invariance), we know that $\langle\phi_p\rangle - \langle\phi_q\rangle$ for any two sites p and q, which leads us to define the generic expectation value $\langle\phi\rangle$

$$\langle\phi\rangle - \langle\phi_p\rangle - \langle\phi_q\rangle - \Phi - \frac{1}{N}\sum_{i\in\Lambda}\langle\phi_i\rangle - \frac{1}{NZ}\mathrm{tr}\left[\left(\sum_{i\in\Lambda}\phi_i\right)e^{-S+J\sum_{i\in\Lambda}\phi_i}\right] \qquad (1.11)$$

Note that $\langle\phi\rangle$ can be obtained directly using the free energy through

$$\langle\phi\rangle = \frac{1}{NZ}\frac{\partial Z}{\partial J} = -\frac{1}{N}\frac{\partial F}{\partial J} = -\frac{\partial f}{\partial J} \qquad (1.12)$$

where, in the last step, we have defined the *free energy density f* by

$$f - \frac{1}{N}F - -\frac{1}{N}\ln Z \qquad (1.13)$$

The free energy density f is an intensive quantity, while the free energy F is extensive. We will later find ourselves preferring calculations with *f*, because the lack of volume dependence makes it a computationally accessible quantity. While discussing *f*, it is interesting to look at $\frac{\partial^2 f}{\partial J^2}$:

$$-\frac{\partial^2 f}{\partial J^2} - \frac{\partial}{\partial J}\left[-\frac{\partial f}{\partial J}\right] - \frac{\partial}{\partial J}\langle\phi\rangle - \chi \qquad (1.14)$$

In the last step we noted the equality with the susceptibility χ (c.f. equation (1.9)). This is easy to confirm by explicitly writing out the left hand side of the above equation.

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Critical Phenomena

Simple magnetic models are the pedagogical cornerstone of statistical mechanics and phase transitions. These models picture magnets as lattices, with the sites occupied by electrons, whose spins contribute to the overall magnetic field generated by the magnet. The spins may be considered as vectors, pointing in any general direction, in which case we are talking about *Heisenberg models.* If we restrict the spins to a few scalar values, we obtain an *Ising model.* In either case, we can define the magnetization, denoted by Φ (c.f. equation (1.10)), a net magnetic field (per lattice site) resulting from the combination of the spins throughout the lattice.

Consider the situation where all the spins point in random directions. We expect the individual spins to cancel, leaving us with $\Phi = 0$, and we say that the system is in the *paramagnetic phase.* Conversely, the system can also be in a state where all the spins are aligned parallel to each other, producing a non-zero magnetization and consigning the magnet to the *ferromagnetic phase.* One important concept being introduced by this discussion is that of an *order parameter.* The knowledge of an order parameter reveals the particular phase a system is in. Specifically, the magnetization is an order parameter for the magnetic system under discussion — if $\Phi = 0$ the system is in the paramagnetic phase, and if $\Phi \neq 0$ the system is in the ferromagnetic phase.

Another important concept is that of a *phase transition.* To illustrate this, we consider (qualitatively) the temperature dependence of our magnet, as shown in figure 1.1. For high temperatures, where we define 'high' as a temperature T greater than some *critical temperature* T_c, the excess thermal energy excites the spins so that they point in random directions, leading to $\Phi \quad 0$. For temperatures $T < T_c$, there is not enough thermal energy to maintain the random oscillations, and the spins tend to settle down into alignment, producing a non-zero Φ. If we imagine the system cooling from a temperature $T > T_c$ to a final temperature $T < T_c$, the system would inevitably have to pass through the point $T = T_c$, which divides the paramagnetic and ferromagnetic phases, or, in other words, defines a phase boundary. We say that the system undergoes a phase transition.

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Phase transitions are, in general, classified according to the Ehrenfest classification. In the scheme, a phase transition is said to be nth *order* if any nth derivative of the free energy with respect to any of its arguments yields a discontinuity at the phase transition [1]. In this context, the phenomena that we will be studying in this thesis

Of particular interest to us is the behaviour of certain physical quantities in the vicinity of the critical point. It was empirically found that, for example, the shape of the magnetization curve for temperatures $T \longrightarrow T_c^-$ can be reproduced by a simple power law. In particular, we can define the *reduced temperature*

$$t - \frac{T_c - T}{T_c} \qquad (1.15)$$

in terms of which we can write:

$$\Phi \sim |t|^{\beta} \qquad (1.16)$$

In the above equation, β is an example of a *critical exponent*, a quantity that we aim to investigate with our LDE approach. A number of other physical quantities exhibit power law behaviour near criticality, and each such power law defines another critical exponent.

Quantity	Behaviour
Magnetization	$\Phi \sim \lvert t\rvert^{\beta}$
Magnetization	$\Phi \sim \lvert J\rvert^{\frac{1}{\delta}}$ at $t-0$
Susceptibility	$\chi \sim \lvert t\rvert^{-\gamma}$

Table 1.1: Definitions of the critical exponents β, δ and γ in terms of the critical behaviour of relevant quantities. Φ is the magnetization, or average field (1.10), J is an external magnetic field, or source (1.2), χ is the susceptibility (1.9), and t is the reduced temperature (1.15).

A remarkable experimental fact about critical exponents is their independence of the underlying system.[2] For example, measurements of the density across the liquid- gas phase transition of

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sulphurhexafluoride and ^{3}He yield the same critical exponents [1]. This leads to the definition of *universality classes*, which are comprised of different systems, all sharing the same set of values for the critical exponents.[3] Another example of two models occupying the same universality class, and of some importance to us (see section 2.1.3), is that of the Ising model and lattice ϕ^4 theory.

Self-Interacting Scalar Field

Briefly, the self-interacting scalar field theory, or simply ϕ^4 theory, is the field theorist's testing ground for just about anything. With a Lagrangean density of

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}\lambda\phi^4 \qquad (1.17)$$

it is relatively simple, but can be used as a zero dimensional toy model for perturbative and non-perturbative expansions (see section 1.4.4), as a one dimensional model of the early expansion of the universe .

as a pedagogical model for introducing spontaneous symmetry breaking, or even as a pedagogical model for introducing the Higgs and the Goldstone[4], and, without exaggeration, the list could go on and on.

In the definition of the ϕ^4 theory (equation (1.17)), m^2 is the mass parameter, and λ is the self-coupling constant. The two parameters play significant roles in our motivation for studying the self-interacting scalar field. On the one hand we can test how well the LDE copes with a large coupling constant, a regime in which standard perturbation theory clearly fails. On the other hand, by changing the sign of the mass parameter, we can study the transition between the fully symmetric phase ($m^2 > 0$) and the broken symmetry phase ($m^2 < 0$), as illustrated by figure 1.1.

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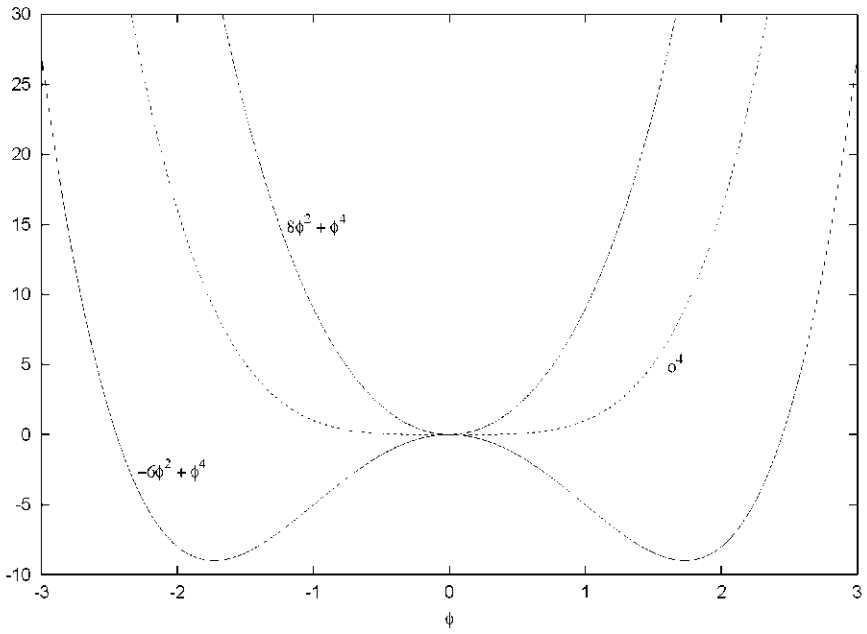

Figure 1.1.: Illustration of the ϕ^4 theory potential, as given by equation (1.17). We set $\lambda = 1$ and alternate between a positive, negative, and zero mass parameter. For $m^2 > 0$, the lowest energy state is at $\phi = 0$. For $m^2 < 0$, the lowest energy state acquires a non-zero value. Although the Lagrangean of the system is Z(2) invariant, the vacuum breaks this symmetry by choosing either of the two lowest energy points. This type of behaviour is called spontaneous symmetry breaking.

The expectation value of the field $\langle\phi\rangle$ is an order parameter of the theory — clearly indicating the broken or unbroken phase by a non-zero or zero value. Actually, we expect the behaviour to be similar to that displayed by a magnetic system, shown in figure 1.1. The difference though, is that the graph would be an expectation value vs. mass parameter plot, rather than the magnetization vs. temperature plot. The power law behaviour, however, is of the same type:

$$\langle\phi\rangle \sim |m^2|^\beta \qquad (1.18)$$

with m^2 approaching m_c^2 from below (from the broken phase). For example, ignoring quantum fluctuations (i.e. in classical ϕ^4 theory), the expectation value of the field for $m^2 < 0$ is of the form:

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$$\langle \phi \rangle \propto \sqrt{-\frac{m^2}{\lambda}} \qquad (1.19)$$

which implies a critical exponent

$$\beta \quad \frac{1}{2} \qquad (1.20)$$

Linear Delta Expansion

The LDE is a general framework which enables us to systematically introduce non- perturbative behaviour into a series expansion by making use of variational parameters. This is achieved by expanding about a soluble approximation for the dynamics where the soluble dynamics contains unphysical parameters. The part that can not be solved directly is then expressed as an expansion about the soluble approximation. The full series, were it available, would give the correct answer (assuming it converges) *and* would be independent of the arbitrary unphysical parameters. However, once truncated, the series will exhibit residual dependence on the unphysical elements. These parameters have to be fixed by some criterion which aims to leave the truncated series as a good approximation to the correct answer. This process of fixing the variational parameters is performed order by order, thus introducing non-perturbative behaviour. In essence, the LDE is a non-perturbative, variational expansion, and as such it should come as no surprise that we find the same methodology hidden behind various different names like optimized perturbation theory [2], modified perturbation expansion [3], optimized expansion [4], screened perturbation theory [5], action-variational approach [6], and the variational cumulant expansion [7].

The LDE has evolved from the less popular *exponential delta expansion* [8, 9], where the action for a scalar ϕ^4 theory is replaced by a term of the form $\phi^{2(1+\delta)}$ and expanded in 5 around the ϕ^2 free theory. The LDE, as outlined in here, has been applied to many problems. A partial list includes the zero dimensional ϕ^4 toy model (which we will use as an example to introduce the method) [10],

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the quantum mechanical anharmonic oscillator [11], U(1) complex field theory on a lattice [12], U(1) complex field theory on a lattice at finite density [13], strong coupling Z(2), U(1) and *SU*(2) lattice gauge theories [14], lattice *SU*(2) Higgs model [15], dynamics of a quantum mechanical slow-roll [16], and dynamics of a quantum field theoretic slow-roll [17].

In this section we will present the formalism of the LDE in general, and then apply it to a simple toy model as an example. Note that this is not an exhaustive review of the subject, but rather an overview aimed at introducing the elements needed for the ensuing study presented in this thesis. In particular, we shall use an action as the basic quantity to apply the LDE, although other quantities like Hamiltonians or quantum mechanical wavefunctions can be used.

We start by considering an action *S* which is complicated enough to prevent us from solving the theory in full. Much like any other series expansion, applying the LDE begins by rewriting the action into a different form. To that end, we introduce a *trial action* S_0, and write

$$S \longrightarrow S_\delta - S_0(\vec{v}) - \delta\big(S_0(\vec{v}) - S\big) \qquad (1.21)$$

We call S_δ the δ-*modified action*, due to the parameter δ which was introduced. The main purpose of δ is that of bookkeeping — the action will be expanded in powers of δ, which will in turn help us keep track of the individual orders of the expansion. Notice that by setting $\delta - 0$ our new action simply becomes $S_{\delta=0} - S_0$, while for $\delta - 1$ we regain our original action, $S_{\delta=1} - S$. This is highly reminiscent of the usual perturbation theory split into a 'free' and 'interacting' part, where the $\delta - 0$ and $\delta - 1$ cases play the role of 'switching' the interaction 'off' and 'on'. What we add in the LDE approach is that we let S_0 depend on some set of non-physical, or rather, *variational* parameters denoted by v. The variational parameters lie at the heart of the LDE method and provide the mechanism through which non-perturbative behaviour is introduced into the model.

The LDE gives us freedom regarding the form of the trial action S_0. In other words, it is for us to choose. However, if it is to be of any practical use, it should conform to a few requirements. The trial action has to

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- itself describe a theory that we can solve.
- be a reasonable approximation to the full theory.
- contain non-physical, variational parameters.

The first two points are rather obvious. Regarding the first one, there would be no use for an S0 that we can not solve for, since in the $\delta - 0$'non-interacting' limit we would have replaced a theory that we can not solve with another that we can not solve. As for the second point, since we are aiming to approximate the full theory with S0, it should be as good an approximation as is permitted by the first requirement. A point related to the second requirement is that we would like the difference S0 — S (equation (1.21)) to be small in some sense, so that upon expansion in powers of δ we can hope for convergence of the series as higher orders are considered. The role of the variational parameters (third requirement) is to introduce non-perturbative behaviour, and will be fully appreciated in the coming sections.

Extracting physics from the action proceeds via the partition function (c.f. equation (1.1)). We introduce the δ-modified partition function Z_δ:

$$Z_\delta = \mathrm{tr}\, e^{-S_\delta} = \mathrm{tr}\left[e^{-S_0(\vec{v})} e^{\delta(S_0(\vec{v})-S)}\right] \tag{1.22}$$

where we have replaced the action S in the original Z by the δ-modified version. The above partition function is nicely set up to be expanded in powers of 5, and we write:

$$Z_\delta - \mathrm{tr}\left[e^{-S_0(\vec{v})} \sum_{n-0}^{\infty} \frac{\delta^n}{n!} \left(S_0(\vec{v}) - S\right)^n\right] - \sum_{n-0}^{\infty} \frac{\delta^n}{n!} Z_n(\vec{v}) \tag{1.23a}$$

$$Z_n(\vec{v}) - \mathrm{tr}\left[e^{-S_0(\vec{v})} \left(S_0(\vec{v}) - S\right)^n\right] \tag{1.23b}$$

where we have defined Zn as the nth order term in the expansion of Z_δ. Since the above expansion has not yet been truncated, with the substitution of $\delta - 1$ the dependence on the variational

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parameters would vanish and we would have the full theory. Notice that $Z_0 = \mathrm{tr}\, e^{-S_0}$, which is a quantity that we can solve for, as we have explicitly chosen S0 to describe a soluble theory. Hence we will also be able to evaluate the other Z_n (for $n \geq 1$), at least in principle. In practice, however, as n grows, so does the complexity of $(S_0 - S)^n$; and at some n = R it will become impractical to go on to calculate the next order. Crucially, even with $\delta = 1$, the truncated expansion will retain residual dependence on the variational parameters. We write

$$Z_\delta^{(R)} \quad Z_\delta^{(R)}(\vec{v}) \quad \sum_{n=0}^{R} \frac{\delta^n}{n!} Z_n(\vec{v}) \qquad (1.24)$$

The superscript in $Z_\delta^{(R)}$ denotes the order at which the expansion is truncated. As it stands, $Z_\delta^{(R)}$ is little more than another perturbative expansion, this time around the perturbation being the difference S0 — S. We have yet to use the variational parameters. Having expanded and truncated the partition function, we are left with $Z_\delta^{(R)}$ depending on the variational parameters $\vec{v}$, as well as the physical parameters of the theory, the coupling constants. In the final stage of the LDE methodology, the variational parameters need to be fixed. Most importantly, this is to be done *order by order* in the expansion, since we would otherwise lose the non-perturbative character of the LDE. If, for example, $\vec{v}$ was to be fixed at some specific order of the expansion and then used for all other orders, we would end up with nothing else than another perturbative expansion. Thus, it is precisely the fact that the optimization is to be carried out at every order which will provide the non-perturbative behaviour of the LDE [21].

The aim is to fix the variational parameters to values which will produce a result closest to the true physical one. The problem in achieving this goal is that there is no unique prescription which tells us how to do this. There are two broad categories of methods used for fixing the variational parameters [2, 22]. These two are the *principle of minimal sensitivity* (PMS) and that of *fastest apparent convergence* (FAC).[5]

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• The PMS method argues that the true partition function depends solely on physical parameters and not on variational ones. Thus, surely our best guess at the true value of Z must be the one where $Z_\delta^{(R)}$ depends least on the variational parameters — at a stationary point. Therefore, we are to search for points where

$$\frac{\partial}{\partial \vec{v}} Z_\delta^{(R)} - 0 \qquad (1.25)$$

At this point, 'small' variations in the components of $\vec{v}$ produce 'negligible' variations in $Z_\delta^{(R)}$, thus we are as close to being independent of the variational parameters as we will ever be.

• The FAC method argues that we should be more concerned with the convergence of the expansion. Thus we should ensure that as we go to higher orders, the Z_n terms contribute less (in some sense) to the total result. Generally, we write

$$Z_\delta^{(R)} - Z_\delta^{(R-r)} - 0 \qquad (1.26)$$

for some chosen value of r in the range $1 \leq r \leq R$.

As stated earlier, the above methods are just a broad classification. The PMS has been adapted and specialized to suit many different variational problems. We will not elaborate further at this stage, since this will be a topic studied in greater detail later (see sections 2.9 and 3.5). With so many ways of fixing the variational parameters, it is easily recognised that choosing the right one can be a tricky matter.

To quote from [18]: "...there are almost as many subtly different criteria for choosing v as there are papers on the linear delta expansion. The main point is to choose one that works."

We note, in passing, the somewhat remarkable fact that the PMS and FAC criteria are in fact tightly related [13]. This relationship will not be described further here, because we are not concerned with the application of the FAC criterion in this study.

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The Partition Function

Consider the integral

$$Z - \int_{-\infty}^{+\infty} dx\, e^{-\sigma x^4} - 2\frac{\Gamma\left(\frac{5}{4}\right)}{\sqrt[4]{\sigma}} \approx 1.81280\,\sigma^{-\frac{1}{4}} \qquad (1.27)$$

As indicated above, we can write down a solution in closed form for Z. For the purposes of this example though, we shall ignore the fact that we know the result and proceed by applying the LDE methodology. From the above equation we identify the action as being the quantity

$$S \quad \sigma x^4 \qquad (1.28)$$

The Expansion The first step is to choose an appropriate trial action, and use it to form the 5-modified action. We choose

$$S_0 = vx^2 \qquad (1.29)$$

$$S_\delta \quad vx^2 - \delta\left(vx^2 - \sigma x^4\right) \qquad (1.30)$$

Note the form of S_0:

- It is very simple to solve. The $\delta - 0$ case defines a simple Gaussian integral.
- We hope that the quadratic will do well at approximating the quartic.
- A variational parameter v is introduced, and is to be fixed by some criterion of choice.

The 5-modified action defines the δ-modified partition function, which is then expanded in powers of δ to get

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$$Z_\delta = \int_{-\infty}^{+\infty} dx\, e^{-vx^2 + \delta(vx^2 - \sigma x^4)} = \sum_{n=0}^{\infty} \left[\frac{\delta^n}{n!} \int_{-\infty}^{+\infty} dx\, e^{-vx^2} \left(vx^2 - \sigma x^4\right)^n \right] \qquad (1.31)$$

We can compare the above expansion to equation (1.23a), and find the general expression for the n^{th} order of the expansion to be

$$Z_n = \int_{-\infty}^{+\infty} dx\, e^{-vx^2} \left(vx^2 - \sigma x^4\right)^n \qquad (1.32)$$

These are Gaussian integrals, easily solved analytically. We give the solutions for the first three orders, including the zeroth:

$$Z_0 = \sqrt{\frac{\pi}{v}} \qquad (1.33a)$$

$$Z_1 = \frac{1}{4}\sqrt{\frac{\pi}{v^5}} \left(-3\sigma + 2v^2\right) \qquad (1.33b)$$

$$Z_2 = \frac{3}{16}\sqrt{\frac{\pi}{v^9}} \left(35\sigma^2 - 20\sigma v^2 + 4v^4\right) \qquad (1.33c)$$

$$Z_3 = \frac{15}{64}\sqrt{\frac{\pi}{v^{13}}} \left(-693\sigma^3 + 378\sigma^2 v^2 - 84\sigma v^4 + 8\sigma^6\right) \qquad (1.33d)$$

We are now in the position to explicitly form the truncated series $Z_\delta^{(R)}$ (c.f. equation (1.24)), with R's in the range from 0 to 3. We also set $\delta = 1$, since it is not needed anymore as a bookkeeping parameter. However, the expansion will exhibit residual dependence on the variational parameter v. To fix v, we will apply the PMS condition, i.e. we will search for stationary points of $Z_\delta^{(R)}$ with respect to v.

Fixing v

To gain a better understanding of the behaviour of the expansion, we have plotted the zeroth, first, second and third orders in figure 1.3. We find that $Z_\delta^{(0)}$ and $Z_\delta^{(2)}$ do not offer good PMS behaviour at

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all — the two quantities actually have no stationary points for real v. This is duly confirmed by taking the derivative of $Z_\delta^{(0)} - Z_0$, given by equation (1.33a):

$$\frac{\partial}{\partial v} Z_\delta^{(0)} - \frac{\partial}{\partial v}\left[\sqrt{\frac{\pi}{v}}\right] - -\frac{1}{2}\sqrt{\frac{\pi}{v^3}} \qquad (1.34)$$

Setting the right hand side of the above equation to zero is obviously fruitless.

We would find similar behaviour for $Z_\delta^{(2)}$ as well. This might seem disturbing at first, but it is actually a generally accepted trend in LDE models [12]. We usually find ourselves working with either odd or even orders only, since the other set will not produce stationary points, i.e. there will be no point which provides a physical basis for fixing the variational parameters.

We now look at $Z_\delta^{(1)}$. By observing the form of the curve in figure 1.3 we can expect a straightforward stationary point (maximum). Mathematically we have

$$Z_\delta^{(1)} \quad Z_0 + Z_1 \quad -\frac{3}{4}\sqrt{\frac{\pi}{v^5}}\left(\sigma - 2v^2\right) \qquad (1.35)$$

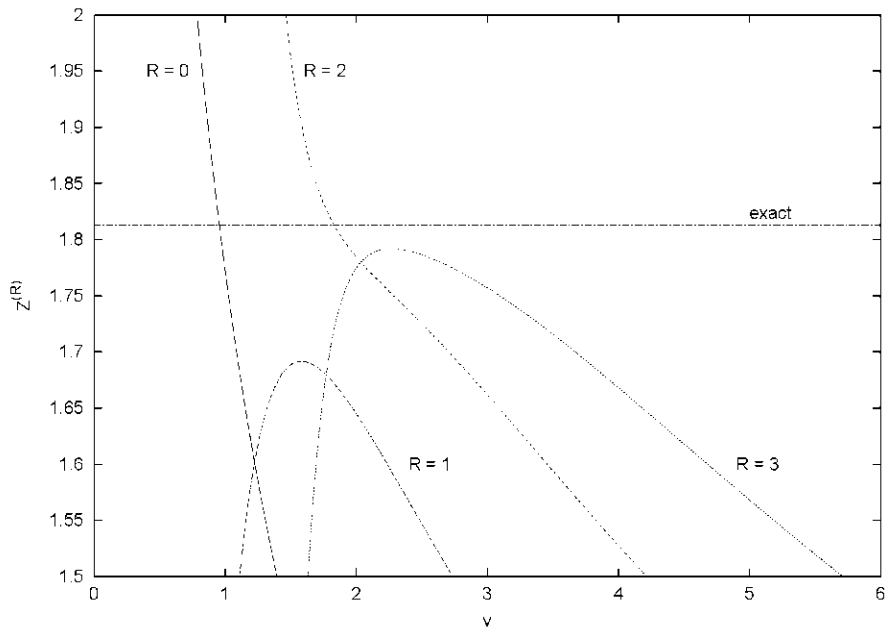

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Figure 1.3: A plot of $Z_\delta^{(R)}$ vs. v for R = 0, 1,2 and 3. We see that only the odd orders give good PMS points. The straight line at 1.81280 is the exact solution of equation (1.27). We set $\sigma = 1$ in this plot.

and, differentiating by v,

$$\frac{\partial}{\partial v} Z_\delta^{(1)} = \frac{3}{8}\sqrt{\frac{\pi}{v^7}}\left(5\sigma - 2v^2\right) \qquad (1.36)$$

We set the above equation equal to zero, and solve for v to get the optimum value which we denote by $\bar{v}$. Substituting back into $Z_\delta^{(1)}$ in equation (1.35), we get our 1st order LDE estimate of Z:

$$v = \sqrt{\frac{5}{2}\sigma} \quad \Longrightarrow \quad Z_\delta^{(1)}(v) \approx 1.69150\,\sigma^{-\frac{1}{4}} \qquad (1.37)$$

We find, to our satisfaction, that the result is quite a good approximation considering that this is only a first order result. In fact, the approximation improves rapidly and already at third order we get to within 1.2% of the true result. Note also the correct functional dependence on the parameter σ.

R	$\bar{v}/\sqrt{\sigma}$	$Z_\delta^{(R)}(\bar{v}) \cdot \sqrt[4]{\sigma}$
1	1.58114	1.69150
3	2.26862	1.79183
5	2.79216	1.80854
7	3.23209	1.81187
9	3.61897	1.81259
∞	—	1.81280

Table 1.2: Summary of the PMS points v and the resulting $Z_\delta^{(R)}(v)$ for all odd orders up to 9. The last row displays the exact result.values of $Z_\delta^{(R)}(\bar{v})$, we see that the expansion is converging towards the exact result. This is also nicely illustrated by the plot in figure 1.4, where we see the maxima of successive odd orders of the expansion converging towards the line denoting the exact result.

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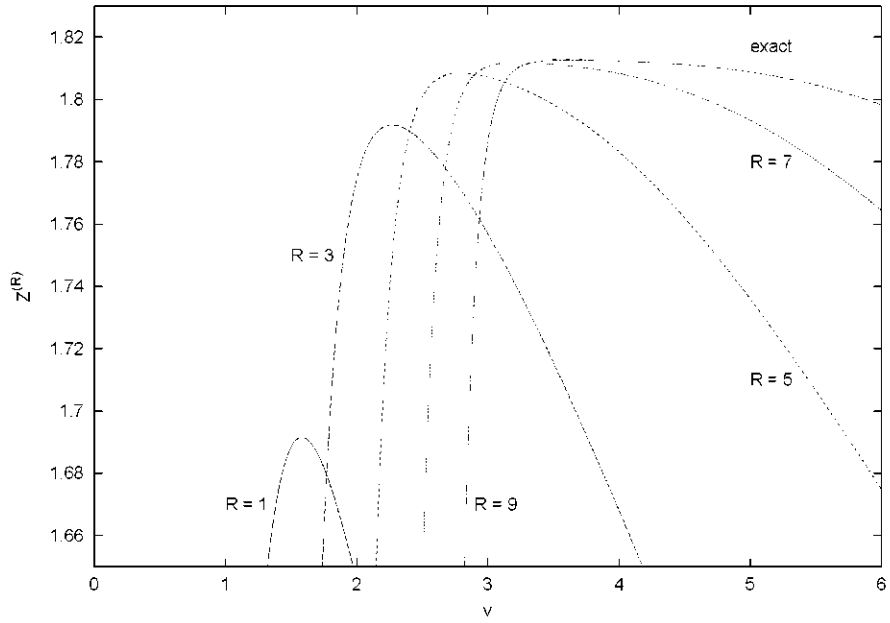

Figure 1.4: A plot of $Z_\delta^{(R)}$ vs. *v* for all odd R up to 9. The maxima of the curves are the appropriate PMS points, and define the value of $\bar{v}$. The straight line at 1.81280 is the exact solution of equation (1.27). We set $\sigma - 1$ in this plot.

Of course, this does not *prove* convergence of our approach. Nevertheless, convergence of the LDE can be proved for this toy model [10]. Moreover, in [21] we find proof of the failure of standard perturbation theory to produce a convergent series for a zero dimensional ϕ^4 theory (a slight generalization of our toy model). The same study defines the general criteria which have to be met by a series to be convergent. These requirements are satisfied by the LDE.

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