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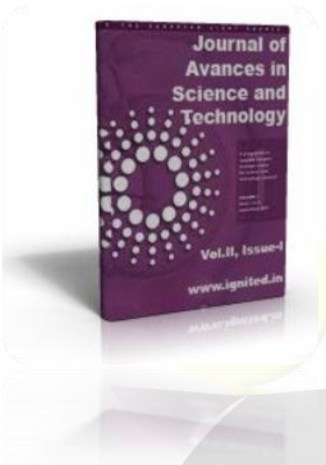
Analysis in Different Phases, Formulas and Conditions of Integer Coefficients Polynomials



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ABSTRACT

We study the problem of minimizing the supremum norm, on a segment of the real line or on a compact set in the plane, by polynomials with integer coefficients. The extremal polynomials are naturally called integer Chebyshev polynomials. Their factors, zero distribution and asymptotics are the main subjects of this paper. In particular, we show that the integer Chebyshev polynomials for any infinite subset of the real line must have infinitely many distinct factors, which answers a question of Borwein and Erdelyi. Furthermore, it is proved that the accumulation set for their zeros must be of positive capacity

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in this case. We also find the first nontrivial examples of explicit integer Chebyshev constants for certain classes of lemniscates.

INTEGER CHEBYSHEV PROBLEM

HISTORY AND NEW RESULTS Define the uniform (sup) norm on a compact set $E \subset \mathbb{C}$ by

$$\|f\|_E := \sup_{z \in E} |f(z)|.$$

The primary goal of this paper is the study of polynomials with integer coefficients that minimize the sup norm on the set E. In particular, we consider the asymptotic behavior of these polynomials and of their zeros. Let $\mathcal{P}_n(\mathbb{C})$ and $\mathcal{P}_n(\mathbb{Z})$ be the classes of algebraic polynomials of degree at most n, respectively with complex and with integer coefficients. The problem of minimizing the uniform norm on E by monic polynomials from $\mathcal{P}_n(\mathbb{C})$ is well known as the Chebyshev problem (see [6], [35], [47], [18], etc.) In the classical case E = [—1,1], the explicit solution of this problem is given by the monic Chebyshev polynomial of degree n:

$$T_n(x) := 2^{1-n} \cos(n \arccos x), \quad n \in \mathbb{N}.$$

Using a change of variable, we can immediately extend this to an arbitrary interval $[a, b] \subset \mathbb{R}$, so that

$$t_n(x) := \left(\frac{b-a}{2}\right)^n T_n\left(\frac{2x-a-b}{b-a}\right)$$

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is a monic polynomial with real coefficients and the smallest uniform norm on [a, b] among all monic polynomials of degree n from $\mathcal{P}_n(\mathbb{C})$. In fact,

$$\|t_n\|_{[a,b]} = 2\left(\frac{b-a}{4}\right)^n, \quad n \in \mathbb{N}, \tag{1.1}$$

and we find that the Chebyshev constant for [a, b] is given by

$$t_{\mathbb{C}}([a,b]) := \lim_{n\to\infty} \|t_n\|_{[a,b]}^{1/n} = \frac{b-a}{4}. \tag{1.2}$$

The Chebyshev constant of an arbitrary compact set $E \subset \mathbb{C}$ is defined in a similar fashion:

$$t_{\mathbb{C}}(E) := \lim_{n\to\infty} \|t_n\|_E^{1/n}, \tag{1.3}$$

where t_n is the Chebyshev polynomial of degree n on E. It is known that $t_{\mathbb{C}}(E)$ is equal to the transfinite diameter and the logarithmic capacity cap(E) of the set E (cf. [47, pp. 71-75], [18] and [34] for the definitions and background material). One may notice that the Chebyshev polynomials on the interval [—2, 2] have integer coefficients. The roots of the n-th Chebyshev polynomial on [—2, 2] are

$$x_k = 2\cos\frac{(2k-1)\pi}{2n}, \quad k = 1, \ldots, n. \tag{1.4}$$

In general, the set of roots of a monic irreducible polynomial over $\mathbb{Z}$ is called a complete set of conjugate algebraic integers. A remarkable result of Kronecker [23] states that any complete set of conjugate algebraic integers, all contained in [—2, 2], must belong to the set of numbers of the form $2\cos(2\pi k/n)$ for all $k \le$ n with

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gcd(k, n) = 1, k,$n \in \mathbb{N}$.Thus we have an exhaustive description of all complete sets of conjugate algebraic integers in [—2, 2]. In fact, Kronecker first proved in [23] that any complete set of conjugates on the unit circle $\{|z| - 1\}$ must be a subset of the roots of unity, and then deduced the above result by using the transformation x = z + 1/z. It is difficult to obtain such a complete characterization when [—2, 2] is replaced by a more general set, but one can extract substantial amount of interesting information from the study of the integer Chebyshev problem (see, e.g., Borwein [5, Ch. 10]):

We say that $Q_n \in \mathcal{P}_n(\mathbb{Z})$ is an integer Chebyshev polynomial for a compact set $E \subset \mathbb{C}$ if

$$||Q_n||_E = \inf_{0 \not\equiv P_n \in \mathcal{P}_n(\mathbb{Z})} ||P_n||_E, \qquad (1.5)$$

where the inf is taken over all polynomials from $P_n(Z)$, that are not identically zero. Note that Q_n may not be unique, and its degree may be less than n. The integer Chebyshev constant (or integer transfinite diameter) for E is given by

$$t_{\mathbb{Z}}(E) := \lim_{n \to \infty} ||Q_n||_E^{1/n}. \qquad (1.6)$$

The existence of the limit in (1.6) follows by the same argument as for (1.3), found in [18] or [47], which also shows that this limit is independent of the choice of a sequence of integer Chebyshev polynomials. It is important that we do not require polynomials to be monic here, as this would lead to a quite

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different problem (cf. Borwein, Pinner and Pritsker [8]). Note that, for any $P_n \in \mathcal{P}_n(\mathbb{Z})$,

$$\|P_n\|_E = \|P_n\|_{E^*},$$

where $E^* := E \cup \{z : \bar{z} \in E\}$, because Pn has real coefficients. Thus the integer Chebyshev problem on a compact set E is equivalent to that on E*, and we can assume that E is symmetric with respect to the real axis $(\mathbb{R}\text{-symmetric})$ without any loss of generality.

One may readily observe that if E = [a, b] and $b - a \geq 4$, then $Q_n(x) \equiv 1,\ n \in \mathbb{N}$,

$$t_{\mathbb{Z}}([a, b]) = 1, \quad b - a \geq 4. \tag{1.7}$$

On the other hand, we obtain directly from the definition and (1.2) that

$$\frac{b-a}{4} = t_{\mathbb{C}}([a, b]) \leq t_{\mathbb{Z}}([a, b]), \quad b - a < 4. \tag{1.8}$$

Hilbert [21] proved an important upper bound

$$t_{\mathbb{Z}}([a, b]) \leq \sqrt{\frac{b-a}{4}}, \tag{1.9}$$

by (1.1) and (1.6), so that by using Legendre polynomials and Minkowski's theorem on the integer lattice points in a convex body. Actually, he worked with the L_2 norm on [a, b], but this gives the same n-th root behavior as the L_∞ norm in (1.6).

With the help of Hilbert's result (1.9), Schur and Polya (see [43]) showed that any interval $[a, b] \subset \mathbb{R}$, of length less than 4, can contain only finitely many complete

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sets of conjugate algebraic integers. Thus one may be able to explicitly find those polynomials with integer coefficients and all roots in [a, b], b — a < 4. These results were generalized to the case of an arbitrary compact set $E \subset \mathbb{C}$ by Fekete [12], who developed a new analytic setting for the problem, by introducing the transfinite diameter of E and showing that it is equal to $t_{\mathbb{C}}(E)$. Both, the transfinite diameter and the Chebyshev constant, were later proved to be equal to the logarithmic capacity cap(E), by Szego [45]. Therefore we state the result of Fekete as follows:

$$t_{\mathbb{Z}}(E) \leq \sqrt{t_{\mathbb{C}}(E)} = \sqrt{\mathrm{cap}(E)}, \quad (1.10)$$

where E is $\mathbb{R}$-symmetric. It contains Hilbert's estimate (1.9) as a special case, since $t_{\mathbb{C}}([a,b]) = (b-a)/4$ by (1.2). Using the same argument as in [43], Fekete concluded by (1.10) that there are only finitely many complete sets of conjugate algebraic integers in any compact set E, satisfying cap(E) < 1. These ideas found many applications, but we only discuss here the developments that are closely related to the subject of this paper. Fekete and Szegő [13] showed that any open neighborhood of the set E, which is symmetric in real axis and has cap(E) = 1, must contain infinitely many complete sets of conjugates. Robinson [36] proved that any interval of length greater than 4 contains infinitely many complete sets of conjugates. But the case of intervals of length exactly 4, or sets of capacity 1 in general, remains open (for further references, see [37], [39], etc.)

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The following useful observation on the asymptotic sharpness for the estimates (1.9) of Hilbert and (1.10) of Fekete is due to Trigub [46].

Remark 1.1. For the sequence of the intervals $I_m := [1/(m + 4), 1/m]$, we have

$$t_{\mathbb{Z}}(I_m) \geq \frac{1}{m+2},$$

so that

$$\lim_{m\to\infty} \frac{t_{\mathbb{Z}}(I_m)}{\sqrt{|I_m|/4}} = 1.$$

We include in Section 5.1 a proof of this fact, due to a relative inaccessibility of the original paper [46].

Remark 1.2. If $\text{cap}(E) \geq 1$ then the problem of evaluating $t_{\mathbb{Z}}(E)$ is trivial, because $\|P_n\|_E \geq (\text{cap}(E))^n$ for any $P_n \in \mathcal{P}_n(\mathbb{Z})$ of exact degree n (cf. [34, p. 155]). This implies that $Q_n \equiv 1$ and $t_{\mathbb{Z}}(E) \quad 1$. Note that $\deg(Q_n) \to \infty$ as $n \to \infty$, unless E is a finite set of points or cap$(E) \geq 1$. We shall exclude these trivial cases from our consideration, by assuming throughout the paper that E is an infinite compact set with cap(E) < 1. This assumption implies that

$$\lim_{n\to\infty} \frac{\deg(Q_n)}{n} = 1,$$

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which is proved in Section 5.1. Hence the definition of the integer Chebyshev constant in (1.6) may be equivalently given as

$$t_{\mathbb{Z}}(E) = \lim_{n\to\infty} \|Q_n\|_E^{1/\deg(Q_n)}.$$

The value of $t_{\mathbb{Z}}([a,b])$ is not known for any segment [a, b], b — a < 4. This represents a difficult open problem, as can be seen from the study of the classical case E = [0,1], which is considered below. From a more general point of view, we are able to find the exact value of $t_{\mathbb{Z}}(E)$ only for a special class of compact sets, namely for lemniscates.

Proposition 1.3. Let

$$V_m(z) := a_m z^m + \ldots + a_0 \in \mathcal{P}_m(\mathbb{Z}),\ a_m \neq 0. \qquad (111)$$

Then we have for the lemniscate

$$L_r := \{z : |V_m(z)| = r\}, \quad 0 \le r < 1, \qquad (1.12)$$

that

$$(r/|a_m|)^{1/m} \le t_{\mathbb{Z}}(L_r) \le r^{1/m}. \qquad (1.13)$$

This gives an immediate corollary.

Corollary 1.4. If $V_m(z)$ of (1.11) is monic, then

$$t_{\mathbb{Z}}(L_r) = r^{1/m}, \qquad (1.14)$$

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where L_r is defined in (1.12). Furthermore, $(V_m)^k$ is an integer Chebyshev polynomial of degree km, $k \in \mathbb{N}$.

One may notice that $t_{\mathbb{Z}}(L_r) = t_{\mathbb{C}}(L_r) = \text{cap}(L_r)$ (see [34, p. 135]) in Corollary 1.4. However, the following result is more interesting.

Theorem 1.5. Suppose that the polynomial $V_m(z)$ of (1.11), with $a_m \in \mathbb{Z}$, is irreducible over integers and that L_r of (1.12) satisfies $r \leq 1/|a_m|$. Then

$$t_{\mathbb{Z}}(L_r) = r^{1/m}, \qquad (1.15)$$

and $(V_m)^k$ is an integer Chebyshev polynomial of degree km, $k \in \mathbb{N}$.

Observe that $t_{\mathbb{Z}}(L_r) \neq t_{\mathbb{C}}(L_r) = \text{cap}(L_r) = (r/|a_m|)^{1/m}$ in this case (cf. [34, p. 135]). Furthermore, we show the sharpness of Fekete's estimate (1.10):

Remark 1.6. For the circle $L_{1/n} = \{z : |nz - 1| = 1/n\}$, $n \in \mathbb{N}$, $n \geq 2$, we have that $t_{\mathbb{Z}}(L_{1/n}) = 1/n$ by Theorem 1.5. On the other hand, $t_{\mathbb{C}}(L_{1/n}) = 1/n^2$, so that equality holds in (1.10).

We note that the above results are also valid for the "filled-in" lemniscates $\{z : |V_m(z)| \leq r\}$, by the maximum modulus principle. A deeper insight into the nature of integer Chebyshev constant and properties of the asymptotically extremal polynomials for integer Chebyshev problem can be found in the study of this problem for E = [0,1]. It was initiated by Gelfond and Schnirelman, who

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discovered an elegant connection with the distribution of prime numbers (see Gelfond's comments in [10, pp. 285-288]). Their argument shows that if $t_{\mathbb{Z}}([0,1]) = 1/e$, then the Prime Number Theorem follows. However, $t_{\mathbb{Z}}([0,1]) > 1/e$, as we shall see below. One can find a nice exposition of this and related topics in Montgomery [24, Ch. 10] (also see Chudnovsky [11]). There is still a chance of success for this approach to the Prime Number Theorem via polynomials in many variables (cf. Chudnovsky [11], Nair [25] and Pritsker [33]).

Proposition 1.7. Let $\mathcal{F}_n \subset \mathcal{P}_n(\mathbb{Z})$ be the set of irreducible over $\mathbb{Z}$ polynomials, of exact degree n, that have all their zeros in a compact set $E \subset \mathbb{C}$. Assuming that

$$s := \liminf_{n\to\infty} \inf_{F_n \in \mathcal{F}_n} c_n^{1/n},$$

where $F_n(x) = c_n x^n + \dots$. Then

$$t_{\mathbb{Z}}(E) \geq 1/s. \qquad (1.16)$$

$\mathcal{F}_n$ is nonempty for an infinite subsequence of $n \in \mathbb{N}$, we define For E = [0,1], Proposition 1.7 coincides with Theorem 2 in [24, p. 182], while the above general form was suggested by the referee. In fact, Montgomery conjectured that equality holds in (1.16) for E = [0,1], but this remains open (essentially the same conjecture was also made in [11, p. 90]). One may try to construct various sequences of polynomials $F_n \in \mathcal{F}_n$, $n \in \mathbb{N}$, to obtain lower bounds for $t_{\mathbb{Z}}([0,1])$

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$$u(x) = \frac{x(1-x)}{1-3x(1-x)},$$

and they give the following lower bound:

$$t_{\mathbb{Z}}([0,1]) \geq 1/s_0 = 0.420726\ldots \qquad (1.17)$$

from (1.16). A few such sequences have been devised (cf. [24] and [11]), with the best known being the Gorshkov sequence of polynomials. It was originally found by Gorshkov in [19], and rediscovered by Wirsing and Smyth. These polynomials arise as the numerators in the sequence of iterates of the rational function (see [24, pp. 183-188]).

Upper bounds for $t_{\mathbb{Z}}([0,1])$ can be obtained directly from the definition of integer Chebyshev constant (1.5)-(1.6). One may even try to find some low degree integer Chebyshev polynomials and compute their norms, only to find out that this is quite a nontrivial exercise. It was noticed in many papers that small polynomials from $\mathcal{P}_n(\mathbb{Z}),\ n \in \mathbb{N},$ arise as products of powers of polynomials from $\mathcal{F}_n,\ k < n.$ Aparicio was the first to prove this in the following strong form (cf. Theorem 3 in [3]):

If a sequence $Q_n \in \mathcal{P}_n(\mathbb{Z}),\ n \in \mathbb{N},$ satisfies

$$\lim_{n\to\infty} \|Q_n\|_{[0,1]}^{1/n} = t_{\mathbb{Z}}([0,1]), \qquad (1.18)$$

then

$$Q_n(x) = (x(1-x))^{[\alpha_1 n]}(2x-1)^{[\alpha_2 n]}(5x^2-5x+1)^{[\alpha_3 n]} R_n(x), \quad \text{as } n \to \infty, \qquad (1.19)$$

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where

$$\alpha_1 \geq 0.1456, \quad \alpha_2 \geq 0.0166 \quad \text{and} \quad \alpha_3 \geq 0.0037, \tag{1.20}$$

and $R_n \in \mathcal{P}_n(\mathbb{Z})$, $\gcd\big(R_n(x), x(1-x)(2x-1)(5x^2-5x+1)\big) = 1$, $n \in \mathbb{N}$.

This gives a good indication of what might be the asymptotic structure of the integer Chebyshev polynomials on [0,1] and other sets. Thus Amoroso [1] considered intervals with rational endpoints, and applied a refinement of Hilbert's approach in [21] to the polynomials vanishing with high multiplicities at the endpoints, to improve upon (1.9). Essentially the same ideas were used by Kashin [22] for dealing with the symmetric intervals [—a, a], for which one should consider polynomials with factors x^k.

Borwein and Erdelyi [7] used numerical optimization techniques to find small polynomials of the form

$$Q_n(x) = \prod_{i=1}^{k} Q_{m_i,i}^{[\alpha_i n]}(x), \quad 0 < \alpha_i < 1,\ i = 1, \ldots, k, \tag{1.21}$$

where $Q_{m_i,i} \in \mathcal{P}_{m_i}(\mathbb{Z})$ and $\sum_{i=1}^{k} \alpha_i m_i - 1$. They improved the upper bound for $t_{\mathbb{Z}}([0,1])$, which triggered a number of numerical studies on the integer Chebyshev polynomials for [0,1] and other intervals. Borwein and Erdelyi also improved the result of Aparicio (1.18)-(1.20):

$$\alpha_1 \geq 0.26,$$

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and used this to show that strict inequality holds in (1.17). Hence the Gorshkov polynomials do not give the exact value of $t_{\mathbb{Z}}([0,1])$.

The ideas of Borwein and Erdelyi have been developed in the papers by Flammang [16], by Flammang, Rhin and Smyth [17], and by Habsieger and Salvy [20], to obtain further numerical improvements in the upper bounds for $t_{\mathbb{Z}}$ on [0,1] and on Farey intervals. In particular, Habsieger and Salvy computed the first 75 integer Chebyshev polynomials for [0, 1] and found the best known upper bound

$$t_{\mathbb{Z}}([0,1]) \leq 0.42347945. \qquad (1.22)$$

Flammang, Rhin and Smyth [17] generalized the approach of [7] to improve the lower bounds in (1.20)

$$\alpha_1 \geq 0.264151, \quad \alpha_2 \geq 0.021963 \quad \text{and} \quad \alpha_3 \geq 0.005285,$$

as well as bounds for six additional factors of the integer Chebyshev polynomials on [0,1]. They also extended the Gorshkov polynomials technique to the Farey intervals [p/q,r/s], with qr — ps = 1, and obtained an interesting generalization of (1.17).

From the above discussion, it is natural to expect that the integer Chebyshev polynomials for [0,1] are built out of the factors as in (1.21), which is suggested in Montgomery [24, p. 182]. In addition, Montgomery proposed studying the zero distribution of these polynomials, along with their associated measures and

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extremal potentials. Potential theory indeed provides powerful methods for dealing with various extremal problems for polynomials, which proved to be very effective for classical Chebyshev polynomials, orthogonal polynomials, etc. It is clear that the study of zeros for integer Chebyshev polynomials is essentially equivalent to the study of their factors and asymptotic behavior. We should note that not all of the zeros of the integer Chebyshev polynomials for [0,1] actually lie on [0,1]. This was discovered by Habsieger and Salvy [20], who found a factor of an integer Chebyshev polynomial of degree 70, with two pairs of complex conjugate roots.

One might hope that the sequence of the integer Chebyshev polynomials for [0,1] is composed from products of powers of a finite number of irreducible polynomials over $\mathbb{Z}$. Unfortunately, this is not true as we show by the following result, answering a question of Borwein and Erdelyi (see [7], Q7).

Theorem 1.8. Let $E \subset \mathbb{R}$ be a compact set, cap(E) < 1, consisting of infinitely many points. Any infinite sequence of the integer Chebyshev polynomials Q_n for E, $n \in \mathbb{N}$, has infinitely many distinct factors with integer coefficients that occur in Q_n's.

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Clearly, if Q_n is irreducible then it is considered a factor of itself. If $E \subset \mathbb{C}$ then the result of Theorem 1.8 may not hold in general, as is shown in Theorem 1.5. It is obvious from the known results that integer Chebyshev polynomials are completely different from the classical companions in their "discrete" nature. However, their zeros cannot be so isolated, as it might appear.

Theorem 1.9. Let $Z \subset \mathbb{C}$ be the set of accumulation points for the zeros of the integer Chebyshev polynomials for a compact set $E \subset \mathbb{R}$, 0 < cap(E) < 1. Then

$$\text{cap}(Z) > 0. \qquad (1.23)$$

This immediately implies that Z cannot be too small, e.g., it cannot be a countable set. One might conjecture that the zeros of the integer Chebyshev polynomials on [0,1] are dense in a Cantor-type set of positive capacity.

Since the nature of the unknown factors of the integer Chebyshev polynomials for [0,1] is rather obscure, we may view the integer Chebyshev polynomials as being of the form

$$Q_n(x) = \left(\prod_{i=1}^{k} Q_{m_i,i}^{l_i(n)}(x) \right) R_n(x), \quad n \in \mathbb{N}, \qquad (1.24)$$

where $l_i(n) \in \mathbb{N}$, $Q_{m_i,i}(x)$ is the known irreducible factor of degree m_i, i = 1,…,k, and $R_n(x)$ is the remaining factor. Assuming that the limits

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$$\lim_{n\to\infty} \frac{l_i(n)}{n} =: \alpha_i > 0, \quad i = 1, \ldots, k, \qquad (1.25)$$

exist, at least along a subsequence, we observe that the n-th root of the absolute value of the product in (1.24) converges to a fixed "weight" function, as $n \to \infty$, locally uniformly in $\mathbb{C}$:

$$\lim_{n\to\infty} \left(\prod_{i=1}^{k} |Q_{m_i,i}(x)|^{l_i(n)} \right)^{1/n} = \prod_{i=1}^{k} |Q_{m_i,i}(x)|^{\alpha_i},$$

where $\sum_{i=1}^{k} \alpha_i m_i \le 1$. Hence, for the purposes of studying the asymptotic behavior, as $n \to \infty$, we may regard $Q_n(x)$ of (1.24) as a "weighted polynomial" and use the methods of weighted potential theory [41]. Following this idea, we generalize the Hilbert-Fekete upper bound for $t_{\mathbb{Z}}$ and find new lower bounds. We also prove various results on the multiplicities of factors and zeros of integer Chebyshev polynomials in the next section. Then we apply the general theory to the integer Chebyshev problem on [0,1] and obtain substantial improvements over the previously known results in Section 3. Section 4 contains a brief outline of the basic facts of weighted potential theory used in this paper. All proofs are given in Section 5.

It must be mentioned that the history of the problem as sketched here is far from being complete. The integer Chebyshev problem is closely connected to approximation by polynomials with integer coefficients (see Ferguson [14] and Trigub [46] for surveys), which has an interesting history of its own. Further

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related topics are: entire functions with integer coefficients (or integer valued) (cf. Polya [29], [30] and [31], Pisot [26], [27] and [28], and Robinson [38], [40], etc.), the integer moment problem (see Barnsley, Bessis and Moussa [4]), the Schur-Siegel trace problem (cf. Schur [43], Siegel [42], Smyth [44], Borwein and Erdelyi [7], Borwein [5], etc.) and many others.

UPPER AND LOWER BOUNDS FOR THE INTEGER CHEBYSHEV CONSTANT

Motivated by the known results on the asymptotic structure of integer Chebyshev polynomials, we study the weighted polynomials $w^n(z)P_n(z)$, where $w(z)$ is a continuous nonnegative function on a compact R-symmetric set $E \subset \mathbb{C}$ and $P_n \in \mathcal{P}_n(\mathbb{Z})$. By analogy with (1.5)-(1.6), consider the weighted integer Chebyshev polynomials $q_n \in \mathcal{P}_n(\mathbb{Z}),\ n \in \mathbb{N},$ such that

$$v_n(E,w) := \|w^n q_n\|_E = \inf_{0 \not\equiv P_n \in \mathcal{P}_n(\mathbb{Z})} \|w^n P_n\|_E,$$

and define the weighted integer Chebyshev constant by

$$t_{\mathbb{Z}}(E,w) := \lim_{n\to\infty} (v_n(E,w))^{1/n}. \qquad (2.1)$$

The limit in (2.1) exists by the following standard argument. Note that

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$$v_{k+m}(E,w) \leq \left\|w^{k+m} q_k q_m\right\|_E \leq \left\|w^k q_k\right\|_E \left\|w^m q_m\right\|_E = v_k(E,w) v_m(E,w).$$

If we set $a_n = \log v_n(E,w)$, then

$$a_{k+m} \leq a_k + a_m, \quad k, m \in \mathbb{N}.$$

Hence

$$\lim_{n\to\infty} \frac{a_n}{n} = \lim_{n\to\infty} \log \left(v_n(E,w)\right)^{1/n}$$

exists by Lemma on page 73 of [47].

Weighted polynomials with complex coefficients are extensively studied in Saff and Totik [41] by means of potential theory. We apply the results of Saff and Totik to the integer Chebyshev problem, and follow their notation and conventions. Our first goal is to give an upper bound for $t_{\mathbb{Z}}(E,w)$. It is possible to generalize the Hilbert-Fekete method for this purpose, but we also need the concept of the weighted capacity of E, denoted by cap(E, w) (see [41] and a brief overview of the weighted potential theory in Section 4).

Theorem 2.1. Let $E \subset \mathbb{R}$ be a compact set and let $w : E \to [0, +\infty)$ be a contin uous function. Then

$$t_{\mathbb{Z}}(E,w) \leq \sqrt{\mathrm{cap}(E,w)}. \qquad (2.2)$$

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Remark 2.2. If $w(z) \equiv 1$ on E then cap(E, 1) = cap(E), so that (2.2) reduces to the result of Fekete (1.10).

It is clear from Section 1 that our main applications are related to the weights of the type

$$w(z) = \left(\prod_{i=1}^{k} |Q_{m_i,i}(z)|^{\alpha_i} \right)^{1/(1-\alpha)}, \qquad (2.3)$$

where the factors $Q_{m_i,i} \in \mathcal{P}_{m_i}(\mathbb{Z})$ have the form

$$Q_{m_i,i}(z) = a_i \prod_{j=1}^{m_i} (z - z_{j,i}), \quad a_i \neq 0, \; i = 1, \ldots, k, \qquad (2.4)$$

and

$$\alpha := \sum_{i=1}^{k} \alpha_i m_i < 1, \qquad (2.5)$$

with $0 < \alpha_i < 1, \; i - 1, \ldots, k$. Thus we readily (see Section 5.2) obtain an upper bound for the classical (not weighted) integer Chebyshev constant.

Theorem 2.3. Suppose that $E \subset \mathbb{R}$ is a compact set, and that the weight w(z) satisfies (2.3)-(2.5). Then

$$t_{\mathbb{Z}}(E) \leq (\text{cap}(E, w))^{(1-\alpha)/2}. \qquad (2.6)$$

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Theorem 2.3 suggests that we may be able to improve the results of Hilbert (1.9) and of Fekete (1.10), by using (2.6) with a proper choice of factors Q_{miii}, i = 1,...,k, for the weight w. It is natural to utilize the known factors of integer Chebyshev polynomials for this purpose. We shall carry out this program in the next section, and obtain an improvement of the upper bound (1.22).

Remark 2.4. After reading the original version of this paper, Chris Smyth drew our attention to the paper of Amoroso [2], where Theorems 2.1 and 2.3 had been proved in equivalent terms, using the concept of f -transfinite diameter. We give a different (and shorter) proof here, using interpolation in weighted Fekete points (see Section 5.2).

It is clear that we need an effective method of finding weighted capacity, in order to make the estimate (2.6) practical. For the "polynomial-type" weights we are considering here, one can express cap(E, w) through the regular logarithmic capacity and Green functions.

Theorem 2.5. Let $E \subset \mathbb{R}$ be a compact set, cap(E) > 0, and let w(z) be as in (2.3)-(2.5). Then there exists a compact set $S_w \subset E \setminus \cup_{i=1}^{k} \{z_{j,i}\}_{j=1}^{m_i}$ such that

$$\operatorname{cap}(E, w) = \exp\left(\int \log w \, d\mu_w - F_w\right), \qquad (2.7)$$

where

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$$F_w = \frac{1}{\alpha - 1}\left(\log \operatorname{cap}(S_w) + \sum_{i=1}^{k} \alpha_i \log|a_i| + \sum_{i=1}^{k}\sum_{j=1}^{m_i} \alpha_i g_\Omega(z_{j,i}, \infty)\right) \tag{2.8}$$

and

$$\mu_w = \frac{1}{1 - \alpha}\left(\omega(\infty, \cdot, \Omega) - \sum_{i=1}^{k}\sum_{j=1}^{m_i} \alpha_i \omega(z_{j,i}, \cdot, \Omega)\right) \tag{2.9}$$

is the unit positive measure supported on S_w. Alternatively,

$$\operatorname{cap}(E, w) = \operatorname{cap}(S_w) \exp\left(\int \log w \, d(\omega(\infty, \cdot, \Omega) + \mu_w)\right). \tag{2.10}$$

Here, $\Omega := \overline{\mathbb{C}} \setminus S_w$, $g_\Omega(z, \xi)$ is the Green function of Ω with pole at $\xi \in \Omega$, and $\omega(\xi, \cdot, \Omega)$ is the harmonic measure at $\xi \in \Omega$ with respect to Ω.

Note that μ_w arises as the equilibrium measure in the weighted energy problem associated with the weight w of (2.3)-(2.5), and F_w is the modified Robin constant for that energy problem (cf. [41] and Section 4 of this paper for the details). The measure $\omega(\infty, \cdot, \Omega)$ is the classical equilibrium distribution on S_w, in the sense of logarithmic potential theory (see [47], [34], etc.)

Using certain information on the asymptotic behavior of integer Chebyshev polynomials, we can find lower bounds for integer Chebyshev constant, as below.

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