[JOURNAL OF ADVANCES IN SCIENCE AND TECHNOLOGY
VOL.-II, ISSUE - I] August 1, 2011
ISSN-2230-9659

# Winning Sets and Hyperbolic Geometry

---


**Vanishree A**
Research Scholar, Singhania University
Rajasthan, INDIA

## REVIEW ARTICLE

Let $H^{d+1} = \{(\upsilon, y) | \upsilon \in R^d, y > \}$,the hyperbolic space of dimension d+1. For d = 1 it is the upper half plane, known also as the Poincar$e'$ upper half plane, consisting of complex numbers with positive imaginary part. $H^{d+1}$ is equipped with the Riemannian metric given by $\frac{1}{y^2}\left(\sum d\upsilon_i^2 + dy^2\right)$ with respect which it is a manifold of constant curvature -1. For $p \in H^{d+1}$ and a unit tangent direction $\xi$ at p the (unit speed) geodesic $\{\gamma(t)\}_{t \in R}$ in $H^{d+1}$ with $\gamma(0) = p$ and $\xi$ as its tangent at t= is a (Euclidean) semicircle in $R^{d+1}$ orthogonal to $R^d$ (the latter is embedded in $R^{d+1}$ as $\{(\upsilon, 0) | \upsilon \in R^d\}$). To the positive segment $\{\gamma(t)\}_{t \geq 0}$ there corresponds a unique $endpo\text{int}\,\gamma(\infty)$ in $R^d$, namely the endpoint of the semicircle as above, in the segment corresponding to the positive part of the geodesic. Conversely, given $p \in H^{d+1}$ and $\upsilon \in R^d$ there exists a unique unit tangent direction $\xi$ at p such that $\upsilon$ is the endpoint of the (positive) geodesic $\{\gamma(t)\}_{t \geq 0}$ such that $\gamma(0) = p$ and $\xi$ is its tangent at 0.

ISSN-2230-9659

Now let M be a manifold of dimension d+1 with constant negative curvature, say -1. Then M has the form $M = \Gamma \backslash H^{d+1}$, where $\Gamma$ is a group of isometries of $H^{d+1}$. The quotient may $\eta : H^{d+1} \to M$, is a covering map, with $H^{d+1}$ as the universal covering space, The action of $\Gamma$ on $H^{d+1}$ extends to an action of $R^d$ as above; the latter may be viewed as the "boundary" of $H^{d+1}$. The geodesics in M consist of the images of geodesics in $H^{d+1}$. Given $q \in M$ and a unit tangent direction $\zeta$ at q and $p \in H^{d+1}$ such that $q = \eta(p)$, the geodesic in M corresponding to q and $\zeta$ has a unique canonical lift in $H^{d+1}$ starting at p which is a geodesic (together with time orientation).

Now suppose furthermore that *M* has finite Riemannian volume. Such a manifold may be compact or noncompact; in the latter case, which is what interests us below, there are finitely many "ends" in the form of tapering off pieces, referred to as "cusps" – given any $\varepsilon > 0$ there exists a compact subset *K* of *M* such that the complement of *M* consists of finitely many subsets, each of which is within (hyperbolic) distance $\varepsilon$ from a geodesic ray (positive time geodesic). It is well-known that for a manifold *M* as above, for almost all pairs $(q,\zeta)$ with $q \in M$ and $\zeta$ a unit tangent direction at *q*, the geodesic $\{\gamma(t)\}_{t\geq 0}$ is dense in *M*, and moreover the pairs $\{(\gamma(t),\gamma'(t))\}$, where $\gamma'(t)$ denotes the tangent to $\gamma$ at *t*, form a dense subset of the unit tangent bundle of *M*. There are however geodesics which are not dense and exhibit a variety of different behaviour. One of the interesting issues in this respect is to understand the set of geodesics with compact closure, when the manifold itself is not compact, but has finite volume. The following result was deduced in this respect.

Corollary 4.1 (Dani) Let *M* be a Riemannian manifold of dimension d+1 with constant negative curvature and finite Riemannian volume. Let $q \in M$. Let $\Phi$ be the set of geodesics $\{\gamma(y)\}_{t\geq 0}$ on *M* such that $\gamma(0) = q$ and the closure of $\{\gamma(y)\}_{t\geq 0}$ in M is compact. Then the set of $\upsilon$ in $R^d$ such that $\upsilon$ is an endpoint of a lift of some $\gamma \in \Phi$ in $H^{d+1}$ is a winning set in $R^d$.

ISSN-2230-9659

In view of the correspondence between the directions and the endpoints of the corresponding geodesic this can also be thought of as a result on the multitude of directions at any point $q \in M$, for which the corresponding geodesic is bounded (has compact closure) in *M*.

## REFERENCES


1. C.S. Aravinda and P. Sankaran, Orbits of certain endomorphisms of nilmanifolds and Housdorff dimension, A tribute to C.S. Seshadri (Chennai, 2002), 125-132.
2. M.B. Bekka and M. Mayer, Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous Spaces, London Mathematical Society Lecture Note Series, 269, Cambridge University Press, Cambridge, 2000.
3. J.W.S. Cassels, An Introduction to Diophantine Approximation, Cambridge University Press, 1957.
4. S.G. Dani, Bounded orbits of flows on homogenous spaces, Comment. Math. Helv. 61 (1986), 636-660.
5. S.G. Dani, On orbits of endomorphisms of tori and the Schmidt game, Ergod. Th. and Dynam. Syst. 8 (1988), 523-529.
6. D. Kleinbock and B. Weiss, Badly approximable vectors on fractals Israel J. Math. 149 (2005), 137-170.
7. W.M. Schmidt, On badly approximable numbers and certain games, Trans. Amer. Math. Soc. 123 (1966), 178-199.
8. W.M. Schmidt, Diophantine Approximation, Springer Verlag 1980.
9. S.G. Dani and G.A. Margulis, Orbit closures of generic unipotent flows on homogeneous spaces of SL(3,IR), Math. Ann. 286 (1990), 101-128.

ISSN-2230-9659

10. P. Walters, An Introduction to Ergodic Theory, Graduate Texts in Mathematics, 79, Springer Verlag, 1982.