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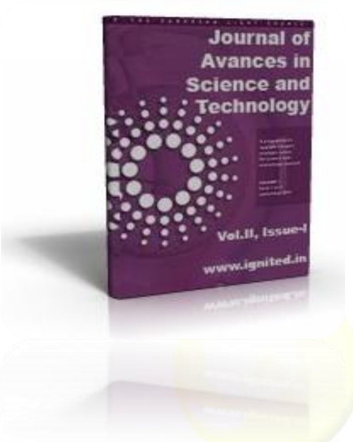
The Gaussian Isoperimetric Inequality, And Its Identified Concentration Phenomenon



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ABSTRACT:-

An isoperimetric inequality of Gaussian sort is determined for the class of probability measures on the Euclidean space, having irritated log-inward densities concerning the standard Gaussian measure.

We audit a few disparities concerning Gaussian measures- isoperimetric inequality, Ehrhard's inequality, Bobkov's inequality, S-inequality what's more correlation conjecture.

Truly, the Gaussian concentration inequality will be the chance to advance some practical logical thoughts around the concentration of measure phenomenon. Specifically, we will perceive how basic semi bunch instruments and the geometry of unique Markov generator may be utilized to study concentration and isoperimetric biases. We investigate in this connection a percentage of the profound associations between isoperimetric imbalances and utilitarian biases of Sobolev sort. We additionally review later take a shot at concentration imbalances in item spaces. Truly, in spite of the fact that the primary subject is Gaussian isoperimetry and examination, numerous thoughts and results have a much broader go of requisitions. We will attempt to demonstrate a portion of the identified fields of investment.

Gaussian favoritisms, whose objective, inexactly talking, comprises of scanning for an inequality between ward (entangled) what's more free (less difficult) structures that turns into a fairness in certain

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(conceivability constraining) cases. We will display a few later conjectures for Gaussian measure/vectors and their accuracy in more level dimension.

The Gaussian isoperimetric inequality, and its identified concentration phenomenon, is a standout amongst the most imperative lands of Gaussian measures. These notes intend to exhibit, in a compact and independent shape, the essential comes about on Gaussian forms and measures dependent upon the isoperimetric instrument. Specifically, our work will incorporate, from this present day perspective, a percentage of the at this point established angles, for example integrability and tail conduct of Gaussian seminorms, huge deviationsor consistency of Gaussian specimen ways. We will additionally focus on a percentage of the later parts of the hypothesis which manage small ball probabilities.

INTRODUCTION

Gaussian random variables and forms dependably assumed a focal part in the probability theory and facts. The cutting edge theory of Gaussian measures joins routines from probability theory, analysis, geometry and topology and is nearly associated with differing requisitions in utilitarian analysis, statistical physics, quantum field theory, budgetary arithmetic and different regions.

In this note we show numerous disparities of geometric nature for Gaussian measures. Every last one of them have rudimentary definitions, yet in any case yield numerous vital and nontrivial results. We start in segment 2 with the recently established Gaussian isoperimetric inequality that motivated in the 70's and 80's the incredible advancement of concentration disparities and their provisions in the geometry and neighborhood theory of Banach spaces. In the spin-off we survey a few later comes about and finalize in area 6 with the discourse of the Gaussian correlation conjecture that remains unsolved more than 30 years.

In the probabilistic requisitions, it is the isoperimetric inequality on circles, as opposed to the established isoperimetric inequality, which is of basic significance. The utilization of the isoperimetric inequality on circles in analysis and probability does a reversal to the new verification, by V. D. Milman, of the well known Dvoretzky hypothesis on circular segments of curved forms [dv]. From that point forward, it has been utilized widely in the neighborhood theory of Banach spaces and in probability theory by means of its Gaussian variant.

Let γ_n denote the standard Gaussian measure on the Euclidean space $\mathbf{R}^n$ with density

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$$\frac{d\gamma_n(x)}{dx} = \frac{1}{(2\pi)^{n/2}}\, e^{-|x|^2/2}, \quad x \in \mathbf{R}^n.$$

The Gaussian isoperimetric inequality states that, for any measurable set $A \subset \mathbf{R}^n$ and any *h* > 0,

$$\gamma_n(A^h) \geqslant \Phi\left(\Phi^{-1}(\gamma_n(A)) + h\right), \qquad (1)$$

where $A^h = \{x \in \mathbf{R}^n : \exists y \in A,\ |x-y| < h\}$

denotes an open /i-neighborhood of A (for the Euclidean distance). Hereinafter, we use the standard notation

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad \Phi(x) = \int_{-\infty}^{x} \varphi(y)\, dy \quad (-\infty \leqslant x \leqslant +\infty),$$

for the marginal density and the marginal distribution function of γ_n with the inverse function $\Phi^{-1} : [0,1] \to [-\infty, +\infty]$

In other words, among all subsets *A* of $\mathbf{R}^n$ with a fixed measure $t = \gamma_n(A)$, the value $\gamma_n(A^h)$ attains minimum for half-spaces of measure *t.*

Letting $h \to 0$ in (1), in the limit one arrives at an equivalent isoperimetric inequality, which may be written as

$$\gamma_n^+(A) \geqslant I(\mu(A)),$$

where $\gamma_n^+(A) = \liminf_{h \to 0} \frac{\gamma_n(A^h) - \gamma_n(A)}{h}$

is the Gaussian perimeter of A, and where

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$$I(t) = \varphi(\Phi^{-1}(t)), \quad 0 \leqslant t \leqslant 1,$$

is the isoperimetric profile (also called the isoperimetric function or the area minimizing function) for the measure γ_n The inequality (1) was uncovered in the mid 1970's freely by Sudakov and Cirefson, and Borell. It has happened to a fundamental importance in the theory of Gaussian random processes, and it is not surprising that for numerous years this result continued to draw in a lot of consideration. These days a few different proofs of (1) are known; given us a chance to mentioned them.

1. The original proof of Sudakov and CireFson and Borell based on the isoperimetric property of balls on the sphere (a theorem due to P. Levy and E. Schmidt).

2. The proof based on the Brunn-Minkowski type inequality due to Ehrhard.

3. The semigroup proof involving Ornstein-Uhlenbeck operators.

4. The proof based on a certain functional form of the isoperimetric inequality on the discrete cube.

5. The proof based on the localization lemma of Lovász–Simonovits.

Some of the developed approaches allowed one to involve in different non-Gaussian probability measures. In particular, as was established by Bakry and Ledoux , one has a similar isoperimetric inequality of Gaussian type

$$\mu(A^h) \geqslant \Phi\left(\Phi^{-1}(\mu(A)) + h\right),$$

for any probability measure μ on $\mathbf{R}^n$., which has a log-concave density with respect γ_n. Equiva- lently, it is the case where μ has density of the form

$$\frac{d\mu(x)}{dx} = e^{-\frac{1}{2}|x|^2 - v(x)}, \quad x \in \Omega,$$

with some convex function $v : \Omega \to \mathbf{R}$, defined on an open convex set Ω in $\mathbf{R}^n$ (bounded or not). On the other hand, Caffarelli showed that any such measure μ represents a contraction of the measure

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γ_n. Hence the inequality for μ, having a log-concave density with respect γ_n, may also be derived from the purely Gaussian case .

A practical Kind of the Isoperimetric Inequality

The isoperimetric property of the Gaussian measure states that for any Borel measurable set $A \subset \mathbb{R}^n$ of measure $\gamma_n(A) = p$ and for all h > 0,

$$\gamma_n(A^h) \geqslant \Phi(\Phi^{-1}(p) + h).$$

Here γ_n is the standard Gaussian measure in $\mathbb{R}^n$, of density

$$d\gamma_n(x) = \prod_{k=1}^n \phi(x_k)\, dx_k,$$
$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n,\ \phi(x_k) = 1/\sqrt{2\pi}\exp(-x_k^2/2)$$

Φ^{-1} is the inverse of the distribution function 0 of γ_1, and A^h = {$x \in \mathbb{R}^n$: $|x - a| < h$ for some $a \in A$} denotes the open h-neighborhood of *A.* becomes identity for all half-spaces *A* of measure p.

In these notes we suggest an equivalent analytic form for above equation involving a relation between smooth functions and their derivatives. Relations of such type are well-known for Lebesgue measure; the Sobolev unequality, for example, provides an equivalent form for the isoperimetric property of balls in the Euclidean space. There is a number of inequalities for the Gaussian measure like Poincare-type or logarithmic Sobolev-type inequalities which can be seen as different versions of so- called "concentration of (Gaussian) measure phenomenon." A question of

interest is whether or not analytic inequalities can contain the isoperimetric inequality (or, its equivalent) as a partial case; the answer is positive.

Theorem. For any smooth function g on $\mathbb{R}^n$ with values in *[0, 1],*

$$\varphi(\Phi^{-1}(\mathbf{E}g)) - \mathbf{E}\varphi(\Phi^{-1}(g)) \leqslant \mathbf{E}\ |\nabla g|.$$

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Conversely, this equation *implies the isoperimetric property of the Gaussian measure.*

As usual, ∇g denotes gradient ofg, and mathematical expectations in equation are understood with respect to measure γ_n

Remark. The function $I(p)=\phi(\Phi^{-1}(p))$ is called the isoperimetric function of γ_n, in the sense that the minimal value of "surface Gaussian measure"

$$\gamma_n^+(A)=\liminf_{\varepsilon\to 0}\frac{\gamma_n(A^\varepsilon)-\gamma_n(A)}{\varepsilon},$$

while $\gamma_n(A)=p$ is fixed, is equal to *I*(*p*). This property, i.e., the inequality

$$\gamma_n^+(A)\geqslant I(p),$$

GAUSSIAN ISOPERIMETRY

For a Borel set *A* in $\mathbb{R}^n$ and $t>0$ let $A_t=A+tB_2^n$ = {$x\in\mathbb{R}^n:|x-a|$ < *t* for some $a\in A$} be the open *t-enlargement* of *A,* where *B*^ denotes the open unit Euclidean ball in $\mathbb{R}^n$. The classical isopcrimctric inequality for the Lcbesguc measure states that if $\mathrm{vol}_n(A)=\mathrm{vol}_n(rB_2^n)$ then $\mathrm{vol}_n(A_t)\geq\mathrm{vol}_n((r+t)B_2^n)$ for *t* > 0. In the early 70's C. Borell and V.N. Sudakov and B.S. Tsirel'son proved independently the isopcrimctric property of Gaussian measures.

Theorem : Let A be a Borel set in $\mathbb{R}^n$ and let H be an affine halfspa.ee such that $\gamma_n(A)=\gamma_n(H)=\Phi(a)$ for some $a\in\mathbb{R}$. Then

$$\gamma_n(A_t)\geq\gamma_n(H_t)=\Phi(a+t)\ \mathit{for\ all}\ t\geq 0.$$

Theorem has an equivalent differential analog. To state it let us define for a measure μ on $\mathbb{R}^n$ and any Borel set *A* the *boundary* μ*-measure* of *A* by the formula $\mu^+(A)=\liminf_{t\to 0+}\frac{\mu(A_t)-\mu(A)}{t}.$

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Moreover let $\varphi(x) = \Phi'(x) = (2\pi)^{-1/2}\exp(-x^2/2)$ and let $I(t) = \varphi \circ \Phi^{-1}(t),\ t \in [0,1]$ *be the* Gaussian isoperimetric function.

The equivalent form of Theorem is that for all Borel sets *A* in $\mathbb{R}^n$

$$\gamma_n^+(A) \geq I(\gamma_n(A)).$$

The equality in above equation holds for any affine halfspace.

For a probability measure μ on $\mathbb{R}^n$ we may define the *isoperimetric function* of μ by

$$\mathrm{Is}(\mu)(p) = \inf\{\mu^+(A) : \mu(A) = p\},\ 0 \leq p \leq 1.$$

Only few cases arc known when one can determine exactly $\mathrm{Is}(\mu)$. For Gaussian measures (2) states that $\mathrm{Is}(\gamma_n) = I$

Let us finish section by an example of application of equation.

Corollary 2.2 Let X be a centered Gaussian random, vector in a separable Banach space $(F, \|\cdot\|)$. Then for any $t > 0$

$$\mathbf{P}(|\|X\| - \mathrm{Med}(\|X\|)| \geq t) \leq 2(1 - \Phi(\frac{t}{\sigma})) \leq e^{-t^2/2\sigma^2},$$

where

$$\sigma = \sup\{\sqrt{\mathbf{E}(x^*(X))^2} : x^* \in F^*, \|x^*\| \leq 1\}.$$

APPROX. GAUSSIAN ISOPERIMETRY FOR K SETS

Consider the canonical Gaussian measure on $\mathbb{R}^n$, γ_n. Given $k \in \mathbb{N}$ and k disjoint measurable subsets of $\mathbb{R}^n$ each of γ_n measure 1/k we can compute the (n — 1)-dimensional Gaussian measure of

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the union of the boundaries of these k sets. Below (see Definition 1) we shall make clear what exactly we mean by the (n — 1)-dimensional Gaussian measure but in particular our normalization will be such that the (n — l)-dimensional Gaussian measure of a hyperplane at distance *t* from the origin will be $e^{-t^2/2}$ (and not $\frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ which is also a natural choice). The question we are interested in is what is the minimal value that this quantity can take when ranging over all such partitions of $\mathbb{R}^n$. As is well known, the Gaussian isoperimetric inequality. The value in question is then 3/2. If the k sets are nice enough (for example if, with respect to the (n — l)-dimensional Gaussian measure, almost every point in the union of the boundaries of the k sets belongs to the boundary of only two of the sets) then the quantity in question is bounded from below by $c\sqrt{logk}$ for some absolute *c* > 0. This was pointed out to us by Elchanan Mossel. Indeed, by the Gaussian isoperimetric inequality, the boundary of each of the sets has measure at least $e^{-t^2/2}$ where *t* is such that $\frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-s^2/2}ds = 1/k$ If k is large enough *t* satisfies

$$\frac{e^{-t^2/2}}{\sqrt{2\pi}2t} < \frac{1}{k} < \frac{e^{-t^2/2}}{\sqrt{2\pi}t}$$

which implies $\sqrt{\log k} \le t \le \sqrt{2\log k}$ and so the boundary of each of the *k* sets has (n — 1)-dimensional Gaussian measure at least $e^{-t^2/2} \ge \sqrt{2\pi}t/k \ge$

$\sqrt{2\pi\log k}/k$. Under the assumption that the sets are nice we then get a lower bound of order $\sqrt{2\pi\log k}$ to the quantity we are after.

Of course the minimality of the boundary of each of the k sets cannot occur simultaneously for even 3 of the k sets (as the minimal configuration is a set bounded by an affine hyperplane) so it may come as a surprise that one can actually achieve a partition with that order of the size of the boundary. To show this is the main purpose of this note. It is natural to conjecture that, for $k-1 \le n$ the minimal configuration is that given by the Voronoi cells of the *k* vertices of a simplex centered at the origin of $\mathbb{R}^n$. So it would be nice to compute or at least estimate well what one gets in this situation. This seems an

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unpleasant computation to do. However, in Corollary 1 below we compute such an estimate for a similar configuration - for even *k* with $k/2 \leq n$, we look

at the *k* cells obtained as the Voronoi cells of $\pm e_i,\ i = 1, \ldots, k/2$ and show that the order of the (n — 1)-dimensional Gaussian measure of the boundary is of order $\sqrt{\log k}$ and we deduce the main result of this note:

Main Result *Given even k with* $k \leq 2n$, *the minimal* (n — 1)*-dimensional Gaussian measure of the union of the boundaries of k disjoint sets of equal Gaussian measure in* $\mathbb{R}^n$ *whose union is* $\mathbb{R}^n$ *is of order* $\sqrt{\log k}.$

In Corollary 2 we deduce analogue estimates for the Haar measure on the sphere S^{n-1}

This note benefitted from discussions with Elchanan Mossel and Robi Krauthgamer. I first began to think of the subject after Elchanan and I.

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