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Applications of Fractional Calculus Operator to Obtaining Certain General Class of Finite Integrals

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ABSTRACT

The object of the this paper are to find two general class of unified finite integrals. We use the technique of Euler integral formula and fractional integral operator in applications. These integrals involve the product of the $\bar{H}$ function, a generalized polynomial set and generalized associated Legendre function of second kind with arguments of the form $(x-a-1)^{-\frac{n}{2}}(x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(gx+f)^{\delta}$. Some special cases and applications are also discussed. Since functions and polynomials occurring in these integrals are general in nature, these results provide interesting unifications and extensions of a large number of new and known results.

Key Words - *$\bar{H}$ function, general class of a multivariable polynomials, generalized associated Legendre function, generalized polynomial set, fractional operator. 2010 hematics Subject Classification – 33E20, 26A33*

1. INTRODUCTION

A large number of integral formulae involving different types of special functions have been developed by many authors. Garg and Mittal [1]obtained an interesting unified integral involving Fox H-function. Considering the work of Garg and Mittal [1], Ali [2]gave three interesting unified integrals involving the hypergeometric function ${}_1F_2$.By using Ali's method [2] Choi and Agarwal [3] presented two generalized integral formulas involving the Bessel function of the first kind , which are expressed in terms of the generalized (Wright) hypergeometric functions.

Agarwal[4] study some new unified integral formulae associated with the $\bar{H}$ -function. Each of these formulae involves a product of the $\bar{H}$ -function and Srivastavapolynomials with essentially arbitrary coefficients. They evaluated the formulae in terms of $\psi(z)$ [logarithmic derivative of $\Gamma(z)$]. Recently Chouhan and Khan [5] presents two new unified integral formulae involving the Fox H-function and M-Series. These results were expressed in terms of the H function.

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2. DEFINITIONS

2.1 Riemann-Liouville Fractional Integral Operator

The Riemann-Liouville fractional integral operator of order ν[6], [7] and [8]is defined by

$$\mathrm{D}_z^{\nu}\{f(z)\} = \begin{cases} \dfrac{1}{\Gamma(-\nu)} \displaystyle\int_0^z (z-t)^{-\nu-1} f(t)\,\mathrm{d}t \text{ , } \mathrm{Re}(\nu) < 0 \\ \dfrac{\mathrm{d}^m}{\mathrm{d}z^m} \mathrm{D}_z^{\nu-m}\{f(z)\} \text{ , } m-1 \leq \mathrm{Re}(\nu) < m \text{ ;} \end{cases} \quad \text{...(2.1)}$$

where m is a positive integer and the integral exists.

2.2 $\bar{H}$ -Function

A more general function known as $\bar{\mathrm{H}}$ -function was introduced by Inayat-Hussain [9] in the following form

$$\bar{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}[z] = \bar{\mathrm{H}}_{\mathrm{P,Q}}^{\mathrm{M,N}}\left[z \left| \begin{matrix} (a_j,\alpha_j;A_j)_{1,\mathrm{N}},(a_j,\alpha_j)_{\mathrm{N+1,P}} \\ (b_j,\beta_j)_{1,\mathrm{M}},(b_j,\beta_j;\mathrm{B}_j)_{\mathrm{M+1,Q}} \end{matrix} \right.\right] = \frac{1}{2\pi \mathrm{i}} \int_{-i\infty}^{i\infty} \bar{\phi}(\xi) z^{\xi}\,\mathrm{d}\xi \quad \text{... (2.2)}$$

Where

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{\mathrm{M}} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{\mathrm{N}} \{\Gamma(1-a_j+\alpha_j \xi)\}^{\mathrm{A}_j}}{\prod_{j=\mathrm{M}+1}^{\mathrm{Q}} \{\Gamma(1-b_j+\beta_j \xi)\}^{\mathrm{B}_j} \prod_{j=\mathrm{N}+1}^{\mathrm{P}} \Gamma(a_j - \alpha_j \xi)} \quad \text{...(2.3)}$$

and $i = \sqrt{-1}$. Here a_j (j = 1,…,P) and b_j (j = 1,…,Q) are complex parameters, $\alpha_j \geq 0 (j = 1,...,\mathrm{P})$ and $\beta_j \geq 0 (j = 1,...,\mathrm{Q})$ and the exponents A_j (j = 1,…,N) and B_j (j = N+1,…,Q) can take any non-integer values .

When allthe exponents A_j and $B_j$takes the value unity, the H-function reduces to the well-known Fox's H-function [10](see also [11]).

Buschman and Srivastava [12]has proved that the integral represented by Eq.(2.2) is absolutely convergent when $\Omega > 0$ and | arg z | < 1/2 $\pi\Omega$, where

$$\Omega = \sum_{j=1}^{\mathrm{M}} \beta_j + \sum_{j=1}^{\mathrm{N}} \mathrm{A}_j \alpha_j - \sum_{j=\mathrm{M}+1}^{\mathrm{Q}} \mathrm{B}_j \beta_j - \sum_{j=\mathrm{N}+1}^{\mathrm{P}} \alpha_j > 0 \quad \text{...(2.4)}$$

The following functions are represented in terms of $\bar{\mathrm{H}}$ function by choosing parameters specifically.

(i) The function connected with certain class of Feynman integrals

$$g[\gamma,\eta,\tau,s;z] = \frac{K_{d-1}\Gamma(s+1)\Gamma\left(\frac{1}{2}+\frac{\tau}{2}\right)}{2^{s+2}\sqrt{\pi}(-1)^s \Gamma(\gamma)\Gamma\left(\gamma-\frac{\tau}{2}\right)} \bar{\mathrm{H}}_{3,3}^{1,3}\left[-z \left| \begin{matrix} (1-\gamma,1;1),\left(1-\gamma+\frac{\tau}{2},1;1\right),(1-\eta,1;1+\rho) \\ (0,1),\left(-\frac{\tau}{2},1;1\right),(-\eta,1;\rho+1) \end{matrix} \right.\right] \text{...(2.5)}$$

Where

$$K_d = \frac{2^{1-d}\pi^{-\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)} \text{ [13]}$$

(ii) The polylogarithm function of order s introducedby Erdelyi et.al. [14] is

$$\mathrm{F}[z, s] = \bar{\mathrm{H}}_{2,2}^{1,2}\left[-z \,\middle|\, \begin{matrix} (1,1;1),\ (1,1;s) \\ (1,1),\ (0,1;s) \end{matrix}\right] \qquad \text{... (2.6)}$$

2.3 Generalized Polynomial Set

The generalized polynomial set $S_n^{\alpha,\beta;\tau}[x]$ is defined by the following Rodrigues typeformula [15]

$$S_N^{\alpha,\beta,\tau}[x] = (Ax+B)^{-\alpha}(1-\tau x^r)^{\beta/\tau}\, T_{k,\ell}^N\left[(Ax+B)^{\alpha+bN}(1-\tau x^r)^{\frac{\beta}{\tau}+\alpha N}\right] \quad \text{...(2.7)}$$

with the differential operator being defined as

$$T_{k,\ell} = x^\ell (k + x\, D_x)$$

Where

$D_x \equiv d/dx$

Raizada [15] presented $S_n^{\alpha,\beta;\tau}[x]$ in the following series form

$$S_n^{\alpha,\beta,\tau}[x] = \sum_{b_1,b_2,a_1,a_2} \theta(b_1,b_2,a_1,a_2)\, x^R (1-\tau x^t)^{sn-a_1} \qquad \text{...(2.8)}$$

where

$$\theta(b_1,b_2,a_1,a_2) = \frac{B^{bn}\,\ell^n(-\tau)^{a_1}(-1)^{b_1}(-a_1)_{a_2}(-b_1)_{b_2}(\alpha)_{b_1}(-\alpha-bn)_{a_2}}{a_1!a_2!b_1!b_2!(1-\alpha-b_1)_{b_2}}$$

$$\cdot\left(\frac{-\beta}{\tau}-sn\right)_{a_1}\left(\frac{b_2+k+ta_2}{\ell}\right)_n\left(\frac{A}{B}\right)^{b_1} \qquad \text{...(2.9)} \quad R = \ell n + b_1 + ta_1 \quad \text{... (2.10)}$$

$$\sum_{b_1,b_2,a_1,a_2} = \sum_{a_1=0}^{n}\sum_{a_2=0}^{a_1}\sum_{b_1=0}^{n}\sum_{b_2=0}^{b_1} \qquad \text{...(2.11)}$$

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2.4 Generalized Associated Legendre Polynomials

Kuipers and Meulenbeld [16] introduced generalized associated Legendre functions

$P_k^{m,n}(z), Q_k^{m,n}(z)$

This function can be presented in terms of hypergeometric function ${}_2F_1(a,b;c;z)$ as

$$Q_k^{m,n}(z) = e^{m\pi i} 2^{k-\frac{m-n}{2}} \frac{\Gamma\left(k+\frac{m+n}{2}+1\right)\Gamma\left(k+\frac{m-n}{2}+1\right)}{\Gamma(2k+2)} \times (z-1)^{-k-\frac{n}{2}-1}(z+1)^{\frac{n}{2}}$$

$$\times {}_2F_1\left(k-\frac{m-n}{2}+1, k+\frac{m+n}{2}+1; 2k+2; \frac{2}{1-z}\right) \qquad \ldots (2.12)$$

$$\text{Where } \left|\frac{2}{1-z}\right| < 1,\ k+m\pm\frac{n}{2} \neq -1,-2,\ldots, 2k+2 \neq 0,-1,-2,\ldots$$

3. INTEGRAL FORMULAE

In this section, two integrals will be evaluated. The integrals are associated with the product of the generalized polynomial set, the $\bar{H}$ -function and generalized associated Legendre functions. The integrals are as follow:

3.1 First Integral

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(gx+f)^{\delta}}{(x-a-1)^{\frac{n}{2}}} S_n^{\alpha,\beta,\tau}\left[y\,\frac{(x-a)^{\sigma}(b-x)^{\eta}}{(cx+d)^{\varsigma}(gx+f)^{\upsilon}}\right]$$

$$\times \bar{H}_{P,Q}^{M,N}\left[z\,\frac{(x-a)^{u}(b-x)^{v}}{(cx+d)^{p}(gx+f)^{q}}\right] Q_k^{m,n}\left(1-\frac{2}{x-a}\right)dx$$

$$= (b-a)^{\lambda+k+\mu}(ac+d)^{\gamma}(bg+f)^{\delta}\, e^{m\pi i}\, 2^{-\left(\frac{m-n}{2}+1\right)}(-1)^{-\left(k+\frac{n}{2}+1\right)}\Gamma\left(k+\frac{m-n}{2}+1\right)$$

$$\times \sum_{b_1,b_2,a_1,a_2}\ \sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3}\Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)} \frac{\tau^{\ell_1} y^{R+\ell_1 t}(a_1-sn)_{\ell_1}}{\ell_1!\ell_2!\ell_3!\ell_4!}$$

$$\times \frac{(b-a)^{\ell_3+(R+t\ell_1)(\sigma+\eta)}\,\theta(b_1,b_2,a_1,a_2)}{(ac+d)^{(R+t\ell_1)\varsigma}(bg+f)^{(R+t\ell_1)\upsilon}}\left(\frac{c(a-b)}{ac+d}\right)^{\ell_2}\left(\frac{g(b-a)}{bg+f}\right)^{\ell_4}$$

$$\times \bar{H}_{P+4,Q+3}^{M,N+4}\left[z\,\frac{(b-a)^{u+v}}{(ac+d)^{p}(bg+f)^{q}}\left|\begin{matrix}\ell_3,(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P}\\(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},\ell_4\end{matrix}\right.\right] \qquad \ldots (3.1)$$

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where

$$L_3 = (1+\gamma-\ell_2-(R+t\ell_1)\zeta, p;1), (1+\delta-\ell_4-(R+t\ell_1)\upsilon, q;1), \\ (-\lambda-k-\ell_3-\ell_2-(R+t\ell_1)\sigma, u;1), (1-\mu-\ell_4-(R+t\ell_1)\eta, v;1) \quad \ldots (3.2)$$

$$L_4 = (1+\gamma-(R+t\ell_1)\zeta, p;1), (1+\delta-(R+t\ell_1)\upsilon, q;1), \\ (-\lambda-\mu-k-(\ell_2+\ell_3+\ell_4)-(R+t\ell_1)(\sigma+\eta), u+v;1) \quad \ldots (3.3)$$

The conditions of validity of Eq (3.1) are

(a) $\mathrm{Re}\,(\lambda,\mu) > 0$

(b) $\min\,\{\sigma, \eta, \zeta, \upsilon, u, v, p, q\} \geq 0$ (not all zero simultaneously)

(c) $1+\mathrm{Re}(\lambda+k)+\mathrm{Re}(\sigma)+u \min\limits_{1\leq j\leq M} \mathrm{Re}\,(b_j/\beta_j) > 0$

$$\mathrm{Re}(\mu)+\mathrm{Re}(\eta)+v \min_{1\leq j\leq M} \mathrm{Re}\,(b_j/\beta_j) > 0$$

(d) $\max\left\{\left|\dfrac{c(b-a)}{ac+d}\right|, \left|\dfrac{g(b-a)}{bg+f}\right|\right\} < 1,\ b \neq a.$

(e) If $k+\dfrac{m+n}{2} \neq -1,-2,\ldots; k\pm\dfrac{m-n}{2} \neq 0,\pm 1,\pm 2,\ldots;$ $2k+2 \neq 0,-1,-2,\ldots$ $|z-1| > 2$,

PROOF. Let L.H.S. of Eq(3.1) is Δ_2.To evaluate the integral, the generalized polynomial set $S_n^{\alpha,\beta,\tau}[z]$ is replaced by its series representation from using Eq(2.8), $\overline{H}$-function is replaced by its Mellin-Barnes contour integral form using Eq(2.2)and $Q_k^{m,n}(z)$ is replaced by its hypergeometric function form using Eq(2.11)in the left hand side ofEq(3.1). Then the powers of (x - a), (b - x), (cx+d) and (gx+f) are collected. In the resulting expression the order of integration and summation is interchanged (which is permissible under the conditions stated with (3.1)) and integral is expressed as follows

$$\Delta_1 = \sum_{b_1,b_2,a_1,a_2} \theta(b_1,b_2,a_1,a_2) \sum_{\ell_1=0}^{\infty} \frac{\tau^{\ell_1} y^{R+\ell_1 t}(a_1-sn)_{\ell_1}}{\ell!} .e^{m\pi i} 2^{-\left(\frac{m-n}{2}+1\right)} (-1)^{-\left(k+\frac{n}{2}+1\right)} \Gamma\left(k+\frac{m-n}{2}+1\right)$$

$$\times \sum_{\ell_3=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3} \Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)\ell_3!}$$

$$\times \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \overline{\phi}(\xi) z^{\xi} \left[\int_a^b (x-a)^{\lambda+k+\ell_3+(R+t\ell_1)\sigma+u\xi} (b-x)^{\mu+(R+t\ell_1)\eta+v\xi-1} (cx+d)^{\gamma-(R+t\ell_1)\zeta-p\xi} \right.$$
$$\left. (gx+f)^{\delta-(R+t\ell_1)\upsilon-q\xi} dx\right] d\xi$$

Taking $\lambda^{\#} = \lambda+k+\ell_3+(R+t\ell_1)\sigma+u\xi,\ \mu^{\#} = \mu+(R+t\ell_1)\eta+v\xi,\ \gamma^{\#} = \gamma-(R+t\ell_1)\zeta-p\xi$

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, $\delta^{\#} = \delta - (R + t\ell_1)\upsilon - q\xi$ and simplifying the powers of (cx+d) and (gx+f) by applying binomial expansions for x $\in$[a, b]

$$(cx+d)^m = (ac+d)^m \sum_{\ell_1=0}^{\infty} \frac{(-m)_{\ell_1}}{\ell_1!} \left\{ \frac{-c(x-a)}{ac+d} \right\}^{\ell_1}, |(x-a)c| < |ac+d|$$

$$(gx+f)^n = (bg+f)^n \sum_{\ell_2=0}^{\infty} \frac{(-n)_{\ell_2}}{\ell_2!} \left\{ \frac{g(b-x)}{bg+f} \right\}^{\ell_2}, |g(b-x)| < |bg+f|$$

$$\Delta_2 = \sum_{b_1,b_2,a_1,a_2} \theta(b_1,b_2,a_1,a_2) \sum_{\ell_1=0}^{\infty} \frac{\tau^{\ell_1} y^{R+\ell_1 t} (a_1 - sn)_{\ell_1}}{\ell!} e^{m\pi i} 2^{-\left(\frac{m-n}{2}+1\right)} (-1)^{-\left(k+\frac{n}{2}+1\right)} \Gamma\left(k + \frac{m-n}{2} + 1\right)$$

$$\times \sum_{\ell_3=0}^{\infty} \frac{\left(k - \frac{m-n}{2} + 1\right)_{\ell_3} \Gamma\left(k + \frac{m+n}{2} + 1 + \ell_3\right)}{\Gamma(2k + 2 + \ell_3)\ell_3!}$$

$$\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \bar{\phi}(\xi) z^{\xi} \left[\int_a^b (x-a)^{\lambda^{\#}} (b-x)^{\mu^{\#}-1} (ac+d)^{\gamma^{\#}} \sum_{\ell_2=0}^{\infty} \frac{(-\gamma^{\#})_{\ell_2}}{\ell_2!} \left\{ \frac{-c(x-a)}{ac+d} \right\}^{\ell_2} \right.$$

$$\left. \times (bg+f)^{\delta^{\#}} \sum_{\ell_4=0}^{\infty} \frac{(-\delta^{\#})_{\ell_4}}{\ell_4!} \left\{ \frac{g(b-x)}{bg+f} \right\}^{\ell_4} dx \right] d\xi$$

The innermost integral is simplified with the help of the Eulerian type integral given by Eq(3.6).

$$\Delta_2 = \sum_{b_1,b_2,a_1,a_2} \theta(b_1,b_2,a_1,a_2) \sum_{\ell_1=0}^{\infty} \frac{\tau^{\ell_1} y^{R+\ell_1 t} (a_1 - sn)_{\ell_1}}{\ell_1!} e^{m\pi i} 2^{-\left(\frac{m-n}{2}+1\right)} (-1)^{-\left(k+\frac{n}{2}+1\right)} \Gamma\left(k + \frac{m-n}{2} + 1\right)$$

$$\times \sum_{\ell_3=0}^{\infty} \frac{\left(k - \frac{m-n}{2} + 1\right)_{\ell_3} \Gamma\left(k + \frac{m+n}{2} + 1 + \ell_3\right)}{\Gamma(2k + 2 + \ell_3)\ell_3!} \sum_{\ell_2,\ell_4} \frac{(-1)^{\ell_2} c^{\ell_2} g^{\ell_4}}{\ell_2!\ell_4!}$$

$$\times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \bar{\phi}(\xi) z^{\xi} (-\gamma^{\#})_{\ell_2} (-\delta^{\#})_{\ell_4} (ac+d)^{\gamma-(R+t\ell_1)\varsigma-\ell_2-p\xi} (bg+f)^{\delta-(R+t\ell_1)\upsilon-\ell_4-q\xi}$$

$$\times (b-a)^{\lambda+k+\ell_2+\ell_3+\ell_4+(R+t\ell_1)(\sigma+\eta)+\mu+(u+v)\xi} B(\lambda^{\#} + \ell_2 + 1, \mu^{\#} + \ell_4) d\xi$$

The beta function is simplified in terms of gamma function and resulting Mellin-Barnes contour integral is interpreted as $\overline{\mathrm{H}}$-function. After little simplification the right hand side of Eq(3.1) is obtained.

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3.2 Second Integral

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}}{(cx+d)^{\gamma}(gx+f)^{\delta}(x-a-1)^{\frac{n}{2}}} S_n^{\alpha,\beta,\tau}\left[y\, \frac{(cx+d)^{\zeta}(gx+f)^{\upsilon}}{(x-a)^{\sigma}(b-x)^{\eta}}\right]$$

$$\times \bar{H}_{P,Q}^{M,N}\left[z\frac{(cx+d)^{p}(gx+f)^{q}}{(x-a)^{u}(b-x)^{v}}\right] Q_k^{m,n}\left(1-\frac{2}{x-a}\right)dx$$

$$=(b-a)^{\lambda+k+\mu}(ac+d)^{\gamma}(bg+f)^{\delta}\, e^{m\pi i}\, 2^{-\left(\frac{m-n}{2}+1\right)}(-1)^{-\left(k+\frac{n}{2}+1\right)}\Gamma\left(k+\frac{m-n}{2}+1\right)$$

$$\sum_{b_1,b_2,a_1,a_2}\ \sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty} \frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3}\Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)} \frac{\tau^{\ell_1} y^{R+\ell_1 t}(a_1-sn)_{\ell_1}}{\ell_1!\ell_2!\ell_3!\ell_4!}$$

$$\times \frac{(ac+d)^{(R+t\ell_1)\zeta}(bg+f)^{(R+t\ell_1)\upsilon}\theta(b_1,b_2,a_1,a_2)}{(b-a)^{(R+t\ell_1)(\sigma+\eta)-\ell_3}}\left(\frac{c(a-b)}{ac+d}\right)^{\ell_2}\left(\frac{g(b-a)}{bg+f}\right)^{\ell_4}$$

$$\times \bar{H}_{P+3,Q+4}^{M+4,N}\left[z\,\frac{(ac+d)^{p}(bg+f)^{q}}{(b-a)^{u+v}}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P},L_3^{*}\\ L_4^{*},(b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q}\end{matrix}\right.\right] \quad \text{... (3.4)}$$

where

$$L_3^{*}=(\gamma-(R+t\ell_1)\zeta,p),(\delta-(R+t\ell_1)\upsilon,q),$$

$$(1+\lambda+\mu+k+\ell_2+\ell_3+\ell_4-(R+t\ell_1)(\sigma+\eta),u+v) \quad \text{... (3.5)}$$

$$L_4^{*}=(\gamma+\ell_2-(R+t\ell_1)\zeta,p),(\delta+\ell_4-(R+t\ell_1)\upsilon,q),$$

$$(1+\lambda+k+\ell_2+\ell_3-(R+t\ell_1)\sigma,u),(\mu+\ell_4-(R+t\ell_1)\eta,v), \quad \text{.. (3.6)}$$

and $\theta(b_1,b_2,a_1,a_2), R$ and $\sum_{b_1,b_2,a_1,a_2}$ are as given in Eqs (2.9), (2.10) and (2.11) respectively.

The conditions of validity for (3.4) are

(a) $\mathrm{Re}(\lambda,\mu)>0$

(b) $\min\{\sigma,\eta,\zeta,\nu,u,v,p,q\} \geq 0$ (not all zero simultaneously)

(c) $1+\text{Re}(\lambda+k)-\text{R}\sigma-u\max_{1\leq j\leq N}\text{Re}\left(\frac{a_j-1}{\alpha_j}\right)>0$

$\text{Re}(\mu)-\text{Re}(\eta)-v\max_{1\leq j\leq N}\text{Re}\left(\frac{a_j-1}{\alpha_j}\right)>0$

(d) $\max\left\{\left|\frac{c(b-a)}{ac+d}\right|,\left|\frac{g(b-a)}{bg+f}\right|\right\}<1,\ b\neq a.$

(e) If $k+\frac{m+n}{2}\neq -1,-2,\ldots;k\pm\frac{m-n}{2}\neq 0,\pm 1,\pm 2,\ldots;$ $2k+2\neq 0,-1,-2,\ldots$

$|z-1|>2$,

PROOF. The integral (3.4) can be evaluated in a similar way as that of the first integral.

4. SPECIAL CASES

Each of our integral formulae (3.1) and (3.4) are unified in nature and possesses manifold generality. On suitably specializing the parameters of the $\bar{H}$ -function, the generalized polynomial set $S_n^{\alpha,\beta,\tau}$ [x] in our main integrals, a large number of new integrals can be obtained as their special cases. one of them are discussed below.

In the third integral reducing $\bar{H}$ function to F(-z, s) function as given by Eq(2.6) and the generalized associated Legendre polynomial $Q_k^{m,n}(x)$ to the associated Legendre polynomial $Q_k^{n}(x)$ as given by Eq(2.12), following result is obtained.

$$\int_a^b \frac{(x-a)^{\lambda-1}(b-x)^{\mu-1}(cx+d)^{\gamma}(gx+f)^{\delta}}{(x-a-1)^{\frac{n}{2}}} S_n^{\alpha,\beta,\tau}\left[y\ \frac{(x-a)^{\sigma}(b-x)^{\eta}}{(cx+d)^{\zeta}(gx+f)^{\upsilon}}\right]$$

$$\times\ F\left[-z\ \frac{(x-a)^{u}(b-x)^{v}}{(cx+d)^{p}(gx+f)^{q}}\right]Q_k^{n}\left(1-\frac{2}{x-a}\right)dx$$

$$=\frac{1}{2}(b-a)^{\lambda+k+\mu}(ac+d)^{\gamma}(bg+f)^{\delta}\ e^{m\pi i}(-1)^{-\left(k+\frac{n}{2}+1\right)}$$

$$\times\sum_{b_1,b_2,a_1,a_2}\ \sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty}\frac{(k+1+\ell_3)\Gamma(k+n+1+\ell_3)}{\Gamma(2k+2+\ell_3)}\frac{\tau^{\ell_1}y^{R+\ell_1 t}(a_1-sn)_{\ell_1}}{\ell_1!\ell_2!\ell_3!\ell_4!}$$

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$$\times \frac{(b-a)^{\ell_3+(R+t\ell_1)(\sigma+\eta)}\,\theta(b_1,b_2,a_1,a_2)}{(ac+d)^{(R+t\ell_1)\zeta}(bg+f)^{(R+t\ell_1)\upsilon}}\left(\frac{c(a-b)}{ac+d}\right)^{\ell_2}\left(\frac{g(b-a)}{bg+f}\right)^{\ell_4}$$

$$\times \bar{H}^{1,6}_{6,5}\left[z\;\frac{(b-a)^{u+v}}{(ac+d)^p(bg+f)^q}\left|\begin{matrix} L_3,\ (1,1;1),\ (1,1;s) \\ (1,1),\ (0,1;s),\ L_4\end{matrix}\right.\right] \qquad \text{... (4.1)}$$

Where L_3 and L_4 are same as given by Eqs(3.5) and (3.6).

5. APPLICATIONS

The results obtained from these integrals can be applied to obtain Riemann-Liouville fractional calculus operator of unified functions. One of the examples is shown below.

Taking b = z, $\eta = v = 0$ in Eq(3.1), the Riemann-Liouville fractional calculus operator of order of a unified function is obtained as

$${}_aD_z^{-\mu}\left\{\frac{(z-a)^{\lambda-1}(cz+d)^{\gamma}(gz+f)^{\delta}}{(z-a-1)^{\frac{n}{2}}}S_n^{\alpha,\beta,\tau}\left[y\;\frac{(z-a)^{\sigma}}{(cz+d)^{\zeta}(gz+1)^{\upsilon}}\right]\right.$$

$$\left.\bar{H}^{M,N}_{P,Q}\left[z^*\;\frac{(z-a)^u}{(cz+d)^p(gz+f)^q}\right]Q_k^{m,n}\left(1-\frac{2}{z-a}\right)\right\}$$

$$= e^{m\pi i}\,2^{-\left(\frac{m-n}{2}+1\right)}(-1)^{-\left(k+\frac{n}{2}+1\right)}\Gamma\left(k+\frac{m-n}{2}+1\right)\Gamma(\mu)\;(z-a)^{\lambda+k+\mu}(ac+d)^{\gamma}(gz+f)^{\delta}$$

$$\sum_{b_1,b_2,a_1,a_2}\;\sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty}\frac{\theta(b_1,b_2,a_1,a_2)\,y^{R+t\ell_1}\tau^{\ell_1}(a_1-sn)_{\ell_1}}{\ell_1!\,\ell_2!\,\ell_3!\,\ell_4!}\;\frac{\left(k-\frac{m-n}{2}+1\right)_{\ell_3}\Gamma\left(k+\frac{m+n}{2}+1+\ell_3\right)}{\Gamma(2k+2+\ell_3)}$$

$$\frac{(-1)^{\ell_2}(z-a)^{\ell_3+(R+t\ell_1)\sigma}(\mu)_{\ell_4}}{(ac+d)^{(R+t\ell_1)\zeta}(gz+f)^{(R+t\ell_1)\upsilon}}\left(\frac{c(z-a)}{ac+d}\right)^{\ell_2}\left(\frac{g(z-a)}{gz+f}\right)^{\ell_4}$$

$$.\bar{H}^{M,N+3}_{P+3,Q+3}\left[\frac{z^*(z-a)^u}{(ac+d)^p(gz+f)^q}\left|\begin{matrix} L_3',(a_j,\alpha_j;A_j)_{1,N},(a_j,\alpha_j)_{N+1,P} \\ (b_j,\beta_j)_{1,M},(b_j,\beta_j;B_j)_{M+1,Q},L_4'\end{matrix}\right.\right] \qquad \text{...(5.1)}$$

where

$$L_3' = (1+\gamma-\ell_2-(R+t\ell_1)\zeta,p;1),\ (1+\delta-\ell_4-(R+t\ell_1)\upsilon,q;1),(-\lambda-k-{}_2-{}_3-(R+t{}_1)\sigma,u;1)$$

$$L_4' = (1+\gamma-(R+t\ell_1)\zeta,p;1),(1+\delta-(R+t\ell_1)\upsilon,q;1),(-\lambda-\mu-k-\ell_2-{}_3-{}_4-(R+t{}_1)\sigma,u;1)$$

and other symbols are same as given in Eqs(2.9), (2.10) and (2.11) respectively. The conditions of validity of Eqs(5.1)can be obtained from those stated with (3.1) and (3.4).

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