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The Algebraic Integer Transfinite Diameter

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Abstract We study the problem of finding nonconstant algebric integer polynomials, normalized by their degree, with small supremum on an interval I. The algebric integer transfinite diameter $t_{\mathrm{M}}(I)$ is defined as the infimum of all such supremums. We show that if I has length 1 then $t_{\mathrm{M}}(I) = \frac{1}{2}$ We make three general conjectures relating to the value of $t_{\mathrm{M}}(I)$ for intervals I of length less that 4. We also conjecture a value for $t_{\mathrm{M}}([0,b])$ where $0 < b \leq 1$. We give some partial results, as well as computational evidence, to support these conjectures. We define functions $L_{-}(t)$ and $L_{+}(t)$, which measure properties of the lengths of intervals I with $t_{\mathrm{M}}(I)$ on either side of t. Upper and lower bounds are given for these functions. We also consider the problem of determining $t_{\mathrm{M}}(I)$ when I is a Farey interval. We prove that a conjecture of Borwein, Pinner and Pritsker concerning this value is true for an infinite family of Farey intervals.

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1. INTRODUCTION

In this paper we continue a study, recently initiated by Borwein, Pinner and Pritsker [2], of the algebraic integer transfinite diameter of a real interval. We write the normalized supremum on an interval I as

$$\|P\|_I^* := \sup_{x \in I} |P(x)|^{1/\deg P}.$$

Note that this is not a norm. Then the algebraic integer transfinite diameter $t_{\mathrm{M}}(I)$ is defined as

$$t_{\mathrm{M}}(I) := \inf_P \|P\|_I^*,$$

where the infimum is taken over all non-constant algebraic polynomials with integer coefficients. We call $t_{\mathrm{M}}(I)$ the algebraic integer transfinite diameter of I (also called the algebraic integer Chebyshev constant [1, 2]). Clearly $t_{\mathrm{M}}(I) \geq \ t_{\mathbb{Z}}(I)$, where $t_{\mathbb{Z}}(I)$ denotes the integer transfinite diameter, defined using the same infimum, but taken over the larger set of all non-constant polynomials with integer coefficients [3, 4, 5]. Further $t_{\mathbb{Z}}(I) \geq \mathrm{cap}(I)$, the capacity or transfinite diameter of I [6, 14], which can be defined again using the same infimum, but this time taken over all non-constant algebraic polynomials with real coefficients. It is well known that $\mathrm{cap}(I) = |I|/4$ for an interval I of length $|I|$. Further, if $|I| \geq 4$ then $t_{\mathbb{Z}}(I) = t_{\mathrm{M}}(I) = \mathrm{cap}(I)$ by [2] so that the challenge for evaluating $t_{\mathrm{M}}(I)$, as for $t_{\mathbb{Z}}(I)$, lies in intervals with $|I| < 4$.

For these intervals we knowfrom [2, Prop. 1.2] that $t_{\mathrm{M}}(I) < 1$. However, in contrast to the study of $t_{\mathbb{Z}}(I)$, in the algebraic case it is possible to evaluate $t_{\mathrm{M}}(I)$ exactly over some such intervals. Our first result is the following.

Theorem 1.1. All intervals I of length **1** have $t_{\mathrm{M}}(I) = \frac{1}{2}$. In fact, slightly more is true: if $1 \leq |I| \leq 1.008848 \ then \ t_{\mathrm{M}}(I) = \frac{1}{2}$.

Furthermore for any b < 1 there is an interval I with $|I| = b \ and \ t_{\mathrm{M}}(I) < \ \frac{1}{2}$, while for h > 1.064961507 there is an interval I with $|I| = b \ and \ t_{\mathrm{M}}(I) > \frac{1}{2}$.

The proof, which is essentially a corollary of Theorem 1.2 (a) below, is discussed in Section 5.

The numbers, 1.008848 and 1.064961507 in Theorem 1.1, like most numerical values given in this paper, are approximations to some exact algebraic number. These numbers are rounded in the correct direction, if necessary, to ensure an inequality still holds. The polynomial equations that they satisfy are given within the text. We have tried to do this for all numerical values.

To measure the range of lengths of intervals having a particular algebric integer transfinite diameter t, we introduce the following two functions:

$$L_-(t) := \inf_I\{|I| : t_M(I) > t\};$$
$$L_+(t) := \sup_I\{|I| : t_M(I) \le t\}.$$

It follows from [2, Prop. 1.3] that both $L_-(t)$ and $L_+(t)$ are nondecreas- ing functions of t. Also $L_-(t) \le L_+(t)$ – see Lemma 3.1(a) below. We give (Proposition 3.1) general method for finding upper and lower bounds for $L_-(t)$ and $L_+(t)$, and apply these methods to get such bounds for $\frac{1}{2} \le t \le 1$. They are constructive, using both the LLL basis-reduction algorithm and the Simplex method. These techniques were first applied in this area by Borwein and Erdelyi [3], and then by Habsieger and Salvy [7]. These bounds are given in Theorem 4.1 and Proposition 4.1 - see also Figures 1 and 2.

At $t = \frac{1}{2}$, we pushed this method further, and were able to say more.

Theorem 1.2. We have

(a) $1.008848 \le L_-\left(\frac{1}{2}\right) \le 1.064961507$

and

(b) $\sqrt{2} \approx 1.41421 \le L_+\left(\frac{1}{2}\right) \le 1.4715.$

Further properties of L_+ and L_- are given in Lemma 3.1.

2. DEFINITIONS, CONJECTURES AND FURTHER RESULTS

In this section, we state some old and some more new results, and (perhaps a little recklessly) make four conjectures.

The following result is simple but fundamental. It is useful for determining lower bounds for $t_M(I)$.

Lemma BPP (Borwein, Pinner and Pritsker [2, p.1905]). Let $Q(x) =$

$a_d x^d + \cdots + a_0$ be a nonmonic irreducible polynomial with integer coefficients, all of whose roots lie in the interval I. Then $\|P\|_I^* \ge a_d^{-1/d}$ for every algebric integer polynomial P, so that $t_M(I) \ge a_d^{-1/d}$. Furthermore, if $\|P\|_I^* = a_d^{-1/d}$ then $t_M(I) = a_d^{-1/d}$ and $|P(\beta)|^{1/\deg P} = a_d^{-1/d}$ for every root β of Q, and Res$(P,Q) = \pm 1$.

The proof follows straight from the classical fact that, for the conjugates β_i of β

$$\mathrm{Res}(P,Q) = a_d^{\deg P} \prod_{i=1}^{d} P(\beta_i)$$

(1)

is a nonzero integer, giving

$$\|P\|_I^* \ge \left(\prod_i |P(\beta_i)|^{1/\deg P}\right)^{\frac{1}{d}} \ge a_d^{-1/d} |\mathrm{Res}(P,Q)|^{\frac{1}{d \deg P}} \ge a_d^{-1/d}.$$

(2)

This result is a variant of a similar one in the theory of $t_{\mathbb{Z}}(I)$—see Lemma 7.1.

We call such a value $a_d^{-1/d}$ in Lemma BPP an obstruction for I, with obstruction polynomial $Q(x)$. From Lemma BPP we see that $t_M(I)$ is bounded below by the supremum of all such obstructions. If this supremum is attained by some value $a_d^{-1/d}$ coming from $Q(x) = a_d x^d + \cdots + a_0$, then we say $a_d^{-1/d}$ is a maximal obstruction, and $Q(x)$ is a maximal obstruction polynomial. It is not known whether such a polynomial exists for all intervals I of length less than 4 (see Conjecture 2.3).

We say that the algebric integer polynomial P(x) is an optimal algebric integer Chebyshev polynomial for I if $\|P\|_I^* = t_{\mathrm{M}}(I)$. If I has a maximal obstruction $a_d^{-1/d}$ with $t_{\mathrm{M}}(I) = a_d^{-1/d}$ and an optimal algebric integer Chebyshev polynomial P then we say that P attains the maximal obstruction $a_d^{-1/d}$.

Throughout this paper, P(x) will denote a algebric integer polynomial, $Q(x)$ a nonmonic integer polynomial and R x any integer polynomial.

One very nice property of the algebric integer transfinite diameter problem, not shared by its nonmonic cousin, is that often exact values can be computed for $t_{\mathrm{M}}(I)$. In all cases where this has been done, including Theorem 1.1, it was achieved by finding a maximal obstruction, and a corresponding optimal algebric integer Chebyshev polynomial. Simple examples of this are given ([2, Theorem 1.5]) by the intervals $I = [0, 1/n]$ for $n \geq 2$, where $Q(x) = nx - 1$ is a maximal obstruction polynomial, and P x x is an optimal algebric integer Chebyshev polynomial. For n= 1, $t_{\mathrm{M}}([0,1]) = \frac{1}{2}$, with Q{x) = 2x— 1 and P(x) =x(x^ 1). This was the case too in [2, Section 5] in the proof of the Farey Interval conjecture for small-denominator intervals.

A much less obvious example is the interval I = [—0.3319,0.7412], of length 1.0731. Here, we have $t_{\mathrm{M}}(I) = \|P\|_I^* = 7^{-1/3} \approx 0.522$, with maximal obstruction polynomial $7x^3 - 7x^2 + 1$ and where P is the optimal algebric integer Chebyshev polynomial

$$\begin{aligned} P(x) = \; & x^{276507}(x-1)^{29858}(x^2+x-1)^{14929} \\ & (x^5-17x^4+24x^3-8x^2-2x+1)^{28848} \\ & (x^7-117x^6+194x^5-70x^4-31x^3+18x^2+x-1)^{7935} \\ & (x^8-4x^7+97x^6-172x^5+78x^4+20x^3-18x^2+1)^{9795} \\ & (x^8-34x^7+164x^6-208x^5+65x^4+33x^3-18x^2-x+1)^{5846} \\ & (x^8-7x^7+2x^6-x^5-10x^4+28x^3-15x^2-2x+2)^{1148} \end{aligned}$$

of degree 670320. (Tighter endpoints for this interval, and its length, can be computed by solving the equation $P(x) = \pm\left(7^{-1/3}\right)^{\deg P}$.) The discovery of this polynomial required the use of Lemma 6.1 below.

For the nonmonic transfinite diameter $t_{\mathbb{Z}}$, Pritsker [13, Theorem1.7] has recently proved that no integer polynomial R x can attain $\|R(x)\|_I^* = t_{\mathbb{Z}}(I)$, this value being achieved only by a normalized product of infinitely many polynomials. An immediate consequence of his result is the following.

Proposition 2.1. If an interval I has an optimal algebric integer Chebyshev polynomial then $t_{\mathrm{M}}(I) > t_{\mathbb{Z}}(I)$.

A fundamental question for both the algebric and nonmonic integer trans- finite diameter of an interval is whether its value can be computed exactly. In [2, Conjecture 5.1], Borwein et al make a conjecture for Farey intervals (intervals $\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right]$ where $b_1, b_2, c_1, c_2 \in \mathbb{Z}$ and $b_2c_1 - b_1c_2 = 1$) concerning the exact value of their algebric transfinite diameter.

Conjecture BPP(Farey Interval Conjecture [1, p. 82], [2, Conjecture 5.1]). Suppose thatis a Farey interval, neither of whose endpoints is an integer. Then $\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right]$

$$t_{\mathrm{M}}\left(\left[\frac{b_1}{c_1}, \frac{b_2}{c_2}\right]\right) = \frac{1}{\min(c_1, c_2)}.$$

Borwein et al verify their conjecture for all Farey intervals having the denominators c_1, c_2 less than 22. In Section 8 we extend the verification to some infinite families of Farey intervals (Theorems 8.2 and 8.3).

We next investigate what happens to $t_{\mathrm{M}}([0,b])$ when b is close to $\frac{1}{n}$.

For these intervals, some surprising things happen. Using the polynomial P(x) = x, we know that $t_{\mathrm{M}}([0,b]) \leq b < \frac{1}{n}$ if $b < \frac{1}{n}$. In fact it appears likely that $t_{\mathrm{M}}([0,b])$, clearly a non-decreasing function of b, has a left discontinu ity at $t = 1/n \quad (n > 1)$. On the other hand, we show in Theorem 9.1 that t_{M} is locally constant on an interval of positive length δ_n to the right of $\frac{1}{n}$. Further, Theorem 9.2 gives much larger values for δ_n for n = 2,3 and 4, as well as an upper bound for δ_2.

In fact, more may be true.

Conjecture 2.1 (Zero-endpoint Interval Conjecture). If $I = [0, b]$ is an in terval with $b \leq 1$, then $t_{\mathrm{M}}(I) = 1/n$, where $n = \max\left(2, \lceil \frac{1}{b} \rceil\right)$ is the smallest integer $n \geq 2$ for which $1/n \leq b$.

What little we know about $t_{\mathrm{M}}([0, b])$ for b 1 is given in Theorem 9.2 (c), (d).

Both Conjecture BPP and Conjecture 2.1 are a consequence of the following conjecture.

Conjecture 2.2 (Maximal obstruction implies $t_{\mathrm{M}}(I)$ Conjecture). If an interval I oflength less than 4 has a maximal obstruction m, then $t_{\mathrm{M}}(I) = m$.

We were at first tempted to conjecture here that $t_{\mathrm{M}}(I)$, as well as equaling its maximal obstruction, is always attained by some algebric integer polynomial. However, the following counterexample eliminates this possibility in general.

Counterexample 2.1. The polynomial $7x^3 + 4x^2 — 2x — 1$ is a maximal obstruction polynomialfor the interval I = [—0.684,0.517]. However, there is nomonic integer polynomial P with $\|P\|_I^*$ equal to the maximal obstruction $7^{-1/3}$ for I.

This result is proved in Section 10.

Our next result proves the existence of maximal obstructions for many intervals.

Theorem 2.1. Every interval not containing an integer in its interior has a maximal obstruction.

Based on Conjecture 2.2 and Theorem 2.1 we make the following conjecture.

Conjecture 2.3 (Maximal Obstruction Conjecture). Every interval of length less than 4 has a maximal obstruction.

We do not have much direct evidence for this conjecture. However, our next conjecture, Conjecture 2.4, implies it. To describe this implication, we need the following notion, taken from Flammang, Rhin and Smyth [5]. An irreducible polynomial $Q(x) = a_d x^d + \cdots + a_0 \in \mathbb{Z}[x]$ with $a_d > 0$, all of whose roots lie in an interval I, and for which $a_d^{-1/d}$ is greater than the (nonmonic) transfinite diameter $t_{\mathbb{Z}}(I)$ is called a critical polynomial for I. Here we are interested only in nonmonic critical polynomials.

It may be that every interval of length less than 4 has infinitely many nonmonic critical polynomials - see Proposition 2.2 below. We make the following weaker conjecture.

Conjecture 2.4 (Critical Polynomial Conjecture). Every interval of length less than 4 has at least one nonmonic critical polynomial.

From Theorem 2.1 below, this conjecture is true for intervals not containing an integer. For intervals I of length less than 4 that do contain an integer (say 0), then, since $t_{\mathbb{Z}}(I) < 1$, the polynomial x is a critical polynomial for I. Thus 'nonmonic' is an important word in this conjecture.

In Theorem 7.1 we prove that Conjecture 2.4 implies Conjecture 2.3. More interestingly, we also prove in Corollary 7.1 that Conjecture 2.2 and Conjecture 2.3 together imply Conjecture 2.4.

We observe in passing the following conditional result for the integer transfinite diameter $t_{\mathbb{Z}}$.

Proposition 2.2. Suppose that an interval I has infinitely many critical polynomials $Q_i(x) = a_{d_i,i}x^{d_i} + \cdots + a_{0,i}$. Then

$$t_{\mathbb{Z}}(I) = \inf_i a_{d_i,i}^{-\frac{1}{d_i}}.$$

This result is proved in Section 7. Montgomery [11, p.182] conjectured this result unconditionally for the interval I 0 1 .

3. UPPER AND LOWER BOUNDS FOR $L_-(t)$ AND $L_+(t)$ FOR FIXED T

The following lemma contains some simple properties, as well as alternative definitions, of L_- and L_+.

Lemma 3.1. We have

(a) $L_-(t) \leq L_+(t)$ for $t \geq 0$;
(b) $L_-(t) = 0$ for $0 \leq t \leq \frac{1}{2}$;
(c) $L_+(t) \geq 2t$ for $0 \leq t \leq 1/2$;
(d) $L_-(t) = \sup_I\{d : t_M(I) \leq t$ for all I with $|I| = d\}$ for $t \geq \frac{1}{2}$;
(e) $L_+(t) = \inf_I\{d : t_M(I) > t$ for all I with $|I| = d\}$ for $t \geq 0$;
(f) $L_+(t) = L_-(t) = 4t$ for $t \geq 1$.

Proof. First note that, by [2, equation (1.11)], $t_M(I) = \frac{1}{2}$ for the zero-length interval $[\frac{1}{2}, \frac{1}{2}]$, from which (b) follows.

Part (c) follows from the fact that $\|x\|^*_{[-t,t]} = t$.

To prove (d), take $t \geq \frac{1}{2}$. Then the set

$$S := \{d : t_M(I) \leq t \text{ for all } I \text{ with } |I| = d\}$$

contains 0 (by (b)), so is nonempty. Put $s = \sup_d S$, and take $d \in S$. Since $I' \subset I$ implies that $t_M(I') \leq t_M(I)$ ([2, Prop. 1.3]), any d' with $0 \leq d' < d$ also lies in S, so that $S = [0, s)$ or $[0, s]$. Hence $L_-(t) \geq s$. On the other hand, for each d s there is an interval I with $|I| = d$ and $t_M(I) > t$. Hence $L_-(t) \leq d$, giving $L_-(t) = s$.

Now (a) follows straight from (b) and (d). The proof of (e), similar to that of (d), is left as an exercise for the reader.

Finally, part (f) follows from the fact that for $|I| \geq 4$ we have $t_M(I) = t_{\mathbb{Z}}(I) = \mathrm{cap}(I) = \frac{|I|}{4}$ (see for instance [2]).

Next, we give a simple lemma, needed for applying Proposition 3.1 below.

Lemma 3.2. Suppose that $I_i = [a_i, b_i]$ {i=1,... ,n) are intervals with $a_1 < a_2 < \cdots < a_n = a_1 + 1$, and put $M := \max_{i=1}^{n-1}(b_{i+1} - a_i), m := \min_{i=1}^{n-1}(b_i - a_{i+1})$. Then

(a) Any interval of length at least M contains an integer translate of some If.

(b) Any interval of length at most m is contained in an integer translate of some I_f.

Proof. Given an interval I of length ℓ, we can, after translation by an integer, assume that $I = [a, b]$, where $a_j \leq a < a_{j+1}$, for some $j < n$.

(a) Suppose that $\ell \geq M$. Then $b_{j+1} \leq a_j + M \leq a + \ell$, so that $[a_{j+1}, b_{j+1}] \subset [a, a+\ell]$.

(b) Suppose that $\ell \leq m$. Then $b_j \geq a_{j+1} + m > a + \ell$, so that $[a, a+\ell] \subset [a_j, b_j]$.

The following proposition will be used to obtain explicit upper and lower bounds for $L_-(t)$ and $L_+(t)$ for particular values of t.

Proposition 3.1.

(a) If $Q(x) = a_d x^d + \cdots + a_0$, with integer coefficients and $a_d > 1$, has roots spanning an interval of length ℓ, then for any $t < a_d^{-1/d}$ we have

$$L_-(t) \leq \ell.$$

(b) Suppose that we have a finite set of polynomials $Q_i(x) = a_{d_i,i}x^{d_i} + \cdots + a_{0,i}$ with all $a_{d_i,i}^{-1/d_i} > t$ with the property that every interval of length ℓ contains an integer translate of the roots of at least one of the polynomials Q_i. Then

$$L_+(t) \leq \ell.$$

(c) Suppose that we have a finite set of intervals I_f such that for each I_f there is a algebric integer polynomial P_f with $\|P_i\|^*_{I_i} \leq t$. Suppose too that every interval of length I is contained in an integer translate of some I_i. Then

$$L_-(t) \geq \ell.$$

(d) If $\|P\|_I^* = t$ for some algebric integer polynomial P and interval I of length , then

$L_+(t) \geq \ell.$

Proof.

a. Given such a $Q(x)$ ℓ and interval I of length ℓ, and $t < a_d^{-1/d}$, then from Lemma BPP we have $t_{\mathrm{M}}(I) \geq a_d^{-1/d} > t$ so that, from the def inition of $L_-(t)$, we have $L_-(t) \leq \ell.$

b. Suppose that every interval I of length ℓ contains some integer translate of the set of roots of some Q_i. Then, by Lemma BPP, $t_{\mathrm{M}}(I) \geq a_{d_i,i}^{-1/d_i} > t$. Hence $t_{\mathrm{M}}(I') > t$ for any interval of length $|I'| \geq \ell$, and so $L_+(t) \leq \ell.$

c. Here, for every interval I of length ℓ with $I + r \subset I_i$ say, (with $r \in \mathbb{Z}$), we have

$$t > \|P_i\|_{I_i}^* \geq \|P_i\|_{I+r}^* = \|P_i(x+r)\|_I^* \geq t_{\mathrm{M}}(I),$$

so thatany I' with $t_{\mathrm{M}}(I') > t$ has $|I'| > \ell$. Hence $L_-(t) \geq \ell$.

d. If $\|P\|_I^* = t$ and $|I| = \ell$ then $t_{\mathrm{M}}(I) \leq t$, so that $L_+(t) \geq \ell$.

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