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SEVERAL SEQUENCE AND NEW CONJECTURES OF ZEROS OF RIEMANN ZETA FUNCTIONS

Several Sequence and New Conjectures of Zeros of Riemann Zeta Functions

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Abstract – It is well known that zeroes of Riemann's zeta function encode a lot of numbertheoretical information, in particular, about the distribution of prime numbers via Riemann's and von Mangpldt's formulas for $\pi(x)$ and $\psi(x)$. The goal of this paper is to present numerical evidence for a (presumably new and not yet proved) method for revealing all divisors of all natural numbers from the zeroes of the zeta function.

This text is essentially a written version of t he talk given by the author at the Department of Mathematics of University of Leicester, UK on June 18, '2012. This talk was based on more intensive computations made after previous author's talk on the same subject given originally at the Mathematical Institute of the University of Oxford on January 26, 2012. The new numerical data indicate that some of conjectures stated in Oxford are, most likely, wrong.

We propose and develop yet another approach to the problem of summation of series involving the Riemann Zeta function $\zeta(s)$, the (Hurwitz's) generalized Zeta function $\zeta(s, a)$, the Polygamma function $\psi^{(p)}(z)$ $(p = 0, 1, 2, \cdots)$, and the polylogarithmic function $\mathrm{Li}_s(z)$. The key ingredients in our approach include certain known integral representations for $\zeta(s)$ and $\zeta(s, a)$. The method developed in this paper is illustrated by numerous examples of closed-form evaluations of series of the aforementioned types; in particular, has been implemented in Mathamatica .Many of the resulting summation formulas are believed to be new.

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INTRODUCTION

In this survey, we will focus on some results related to the explicit location of zeros of the Riemann zeta function.

We have few techniques to know the exact behavior of zeros of zeta functions. For the author's study 011 zeros of zeta functions, the Riemann-Siegel formula. We have collected several examples whose structures look like the Riemaim-Siegel formula and which satisfy the analogue of the Riemann hypothesis. The Riemann-Siegel formula can be simplified by

$$f(s) + \overline{f(1-\bar{s})},$$

where $f(s)$ satisfies certain nice conditions. Due to the symmetry, zeros of this formula tend to lie on $\mathrm{Re}(s) = 1/2$. Thus, our strategy to the Riemann hypothesis

(RH) is to find a nice representation of the Riemann zeta function satisfying the above formula: if we are able to prove that complex zeros of $f(s)$ are in $\mathrm{Re}(s) \geqslant 1/2$ or in $\mathrm{Re}(s) \leqslant 1/2$, then we essentially derive RH.

For $T > 0$, we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T),$$

where $N(T)$ is the number of zeros of the Riemann zeta function in $0 < \mathrm{Im}(s) < T$. Thus, the average gap of consecutive zeros of the Riemann zeta function in $0 < \mathrm{Im}(s) < T$ is

$$\frac{2\pi}{\log T}.$$

What can we say about gaps of zeros of the Riemann zeta function? This question is one of the most important questions in studying the behavior of zeros of the Riemann zeta function. Together with the Riemann-Siegel formula. Montgomery's pair correlation conjecture should be realized as the intrinsic property for the behavior of zeros of the Riemann zeta function.

Montgomery's Pair correlation conjecture (PCC) *Assume the Riemann hypothesis. Then we have*

$$\sum_{\substack{0<\gamma,\gamma'<T \\ \frac{2\pi\alpha}{\log T}<\gamma'-\gamma<\frac{2\pi\beta}{\log T}}} 1 \sim N(T)\int_\alpha^\beta 1-\left(\frac{\sin \pi x}{\pi x}\right)^2 dx$$

where $1/2+i\gamma,\ 1/2+i\gamma'$ *are zeros of the Riemann zeta function.*

ZEROES OF SUMS

ZEROES OF PARTIAL SUMS:-

Since the series does not converge for $\sigma \leq 1$, it it is difficult to picture, on the face of it, what relationship, if any, would exist between the zeros of the *truncated zeta-function*

$$\zeta_N(s) := \sum_{n=1}^{N} \frac{1}{n^s}$$

and $\zeta(s)$ in this half plane. The following plot, made possible through the application of the zero finder, illustrates a few points. What is first noticeable is a string of zeros of ζ_{211} near the critical line $\sigma = 1/2$.

$$\begin{aligned}\zeta(s) &= \zeta_{N-1}(s) + N^{-s} + \frac{N^{1-s}}{s-1} + \\ &+ \sum_{n=1}^{\infty} \frac{B_{2n}}{(2n)!}\left(\prod_{j=0}^{2n-2}(s+j)\right) N^{1-s-2n} + R\end{aligned}$$

implies that $\zeta(s)$ is roughly approximated by $\zeta_N(s)$ near the critical line for $t < 2\pi N$, but t also large enough so that $N^{1/2}/t$ is 'small'. The strip of zeros is in accordance with Spira's observation. Above $t = 2\pi N$, about 1326, in Figure 1, the zeros scatter more wildly.

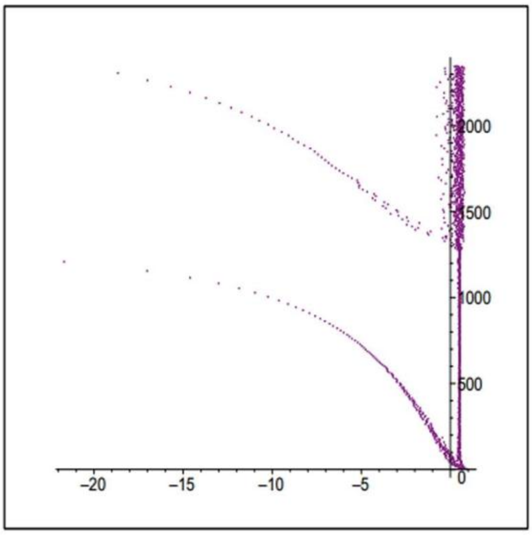

Fig. 1 2000 zeros of $\zeta_{211}(s)$

ZEROES OF SMALL SUMS:-

Single parameter curves. Figure 1 shows the positions of the first '2000 zeros of

$$\zeta_3(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s}$$

in the upper half plane.

Although the zeros are not recurring in a completely regular pattern, their positioning does appear to have a semi periodic nature. The algorithm used to locate these zeros used a homotopy. In its more general form, for ζ_N, starting from the known position of the zeros of the end terms

$$1 + \frac{1}{N^s},$$

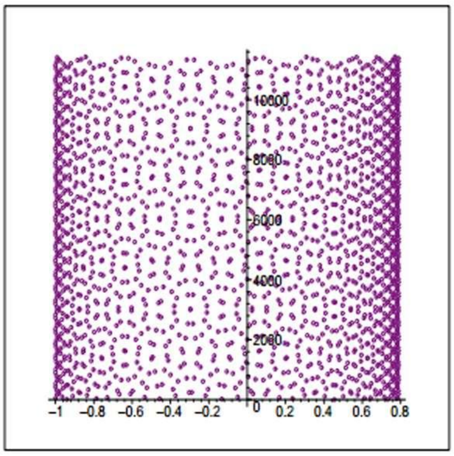

Fig.2 First 2000 zeros of $\zeta_3(s)$

at the points $\{2\pi ki/\log(N) : k \in \mathbb{N}\}$, Newton's method was used to find zeros along the pat

$$1+\frac{1}{N^s}+t\left(\sum_{n=2}^{N-1}\frac{1}{n^s}\right)$$

as t increased from 0 to 1. That this method worked so well in locating all of the zeros of $\zeta_N(s)$ up to heights tested, suggests that the error estimate in above could be improved to $O(1)$. In any case, in such finite exponential sums, the largest integer in the expansion, *N,* is an indicator or the number of zeros to be found up to a height *T* in much the same way as the degree of a polynomial determines its number of zeros in the complex plane.

GENERALITIES

Let us recall that zeta vanishes at negative odd integers. These zeros are called the *trivial zeros* of $\zeta(s)$. The functional equation

$$\zeta(s) = \chi(s)\zeta(1-s), \qquad \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \quad (1)$$

entails that other zeros of $\zeta(s)$ (they are called the *non-trivial zeros)* are symmetric with respect to the critical line $\Re(s) = 1/2$: for each non trivial zero $s = \sigma + it,$

the value $s' = 1 - \sigma +$ *it* is also a zero of $\zeta(s).$

The non-trivial zeros lie in the critical strip

We show that all the non-trivial zeros of $\zeta(s)$ lie in the *critical strip* defined by values of the complex number *s* such that $0 < \Re(s) < 1.$ Because of the functional equation, it suffices to show that does not vanish on the closed half plane $\Re(s) \geq 1.$

The Euler infinite product

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}},$$

valid for all complex numbers .s with $\Re(s) > 1$, shows that $\zeta(s)$ does not vanish

for $\Re(s) > 1$ (a convergent infinite product cannot converge to zero because its logarithm is a convergent series). Thus it suffices now to prove that $\zeta(s)$ does not vanish on the line $\Re(s) = 1.$

This property is in fact the key in the proof of the prime number theorem, and Hadamard and De La Vallée Poussin obtained this result independently in 1896 by different mean (this problem is in fact a first step in a determination of a zero-free region, important to obtain good error terms in the prime number theorem). We present here the argument of De La Vallée Poussin which Is simpler to expose and more elegant.

The Zeta-function has no zeros on the line $\Re(s) = 1$

The starting point is the relation

$$3 + 4\cos\phi + \cos 2\phi = 2(1+\cos\phi)^2 \geq 0 \quad (2)$$

for all values of the real number ϕ. The Euler infinite product writes as

$$\zeta(s) = \exp\left(\sum_{p \text{ prime}}\sum_{m=1}^{\infty}\frac{1}{mp^{ms}}\right), \qquad \Re(s) > 1$$

thus for the complex number $s = \sigma + it$ we have

$$|\zeta(s)| = \exp\left(\sum_{p \text{ prime}}\sum_{m=1}^{\infty}\frac{\cos(mt\log p)}{mp^{m\sigma}}\right), \qquad \sigma = \Re(s) > 1.$$

This relation entails

$$\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| = \exp\left(\sum_{p \text{ prime}}\sum_{m=1}^{\infty}\frac{3+4\cos(mt\log p)+\cos(2mt\log p)}{mp^{m\sigma}}\right)$$

so with (2) we deduce

$$\zeta(\sigma)^3|\zeta(\sigma+it)|^4|\zeta(\sigma+2it)| \geq 1, \qquad \sigma > 1. \quad (3)$$

Now suppose that $1+it$ Is a zero of $\zeta(s)$. Letting $\sigma \to 1$, we have $\zeta(\sigma) \sim 1/(\sigma-1)$

and $\zeta(\sigma+it) = O(\sigma-1)$, so that $\zeta(\sigma)^3|\zeta(\sigma+it)|^4 = O(\sigma-1)$ thus as $\sigma \to 1$, (3)

entails that $|\zeta(\sigma+2it)|$ tends to infinity, which is impossible since $\zeta(s)$ is analytic

around $1+2it$. Thus we have proved that $\zeta(s)$ has no zeros on the line $\Re(s) = 1$.

Other proofs of this result can be found in [3].

CONCLUSION

The present thesis deals with the Riemann zeta-function $\zeta(s)$ and it is intended ad an exposure of some of the most relevant results obtained until today. The zeta-function plays a fundamental role in number theory and one of the most important open question in mathematics is the Riemann Hypothesis (RH), which states that all non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$. The zeta-function appeared for the first time in 1859 on a Riemann's paper originally devoted to the explicit formula connecting the prime counting function, $\pi(x)$, with the logarithmic integral $\mathrm{Li}(x)$; nevertheless, the paper contained other outstanding results, like the analytical continuation of $\zeta(s)$ through the whole complex plane, and deep conjectures, each of them involving the non-trivial zeros of $\zeta(s)$. All these conjectures, thanks to von Mangoldt and Hadamard, afterwards became theorem except, as said before, the conjecture about the displacement of non-trivial zeros along the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

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