"A Structure Theorem for Multiplicative Functions over the Gaussian Integers and Application"

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Abstract – In this paper we present about to show a structure theorem for multiplicative functions on the Gaussian integers which perform that each delimited multiplicative purpose on the Gaussian integers can be decayed addicted to a term which is about periodic and one more which has a small U ^{3} - Gowers uniformity norm.

Keywords: Theorem, Structure, Multiplicative Functions Integers

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INTRODUCTION:

Structure theory in the finite setting - The structure theorem for functions on Z d is a significant tool in preservative combinatory. It has been deliberate broadly in [1-9]. Roughly speaking, the structure theorem says that every function f can be decomposed into one part with a good uniformity property, meaning it has a small Gowers norm, and another with a good structure, meaning it is a nilsequence with bounded complexity. A natural question to ask is: can we get a better decomposition for functions f satisfying special conditions? For example, Green, Tao and Ziegler [5-7] gave a refined decomposition result for the von Mangoldt function Λ. They showed that under some modification, one can take the structured part to be the constant 1.

The Gaussian integers

Z[i] and its norm - Real quadratic ring Z[√ n] (meaning Z[√ n] ⊆ R) may accustomed to resolve Pell's equation x 2 − ny2 = 1 since, over Z[√ n], $x^2 - ny^2$ factors as $x^2 - ny^2$ = (x + y √ n)(x − y √ n). Specially, we defined the conjugate of an element α = x + y √ n to be α = x − y √ n, and then the norm to bet N: Z [√ n] → Z by N (α) = αα = $x^2 - ny^2$. So the solutions to Pell's equation are exactly the elements of norm 1 in Z [√ n], which we showed form a group under multiplication that is generated by two rudiments € and −1. Correspondingly if one needs to study the equation $x^2 + ny^2 = k$ it makes sense to look at the fantasy quadratic ring Z [√ −n] (meaning Z [√ −n] ⊆ C and Z [√ −n] 6⊆ R). The fantasy quadratic rings can be treated in the same basic way as the real quadratic rings theoretically, however their flavor is quite different. For example $x^2 + ny^2 = 1$ has only finitely many solutions, but $x^2 - ny^2 = 1$ has infinitely many (for n nonsquare). For the time being, we will immediately treat the simplest case, the Gaussian integers, which were first studied in detail by Gauss. This ring is related to questions about Pythagorean triples, and more usually, which numbers are sums of two squares.

Definition:

The Gaussian integers are the ring

Z[i] = {a + bi: a, b £ Z}.

For α = a + bi, the conjugate of α is α = a − bi, and the norm is N (α) = |α| 2 = αα = a 2 + b 2. Note that if we draw α as a vector in the complex plane, c = |α| denotes the length of this vector, so the norm of α is just the square of the length of the vector α, i.e., N (α) = c 2. Hence the formula for the norm is precisely the Pythagorean Theorem: αα = a 2 + b 2 = c 2 = N (α).

The reason we want the norm to be the square of the length, instead of just the length is because c 2 is always an integer, but c rarely is, e.g., 1 2 + 12 = √ 2 2 . With this definition, the norm is a map from the ring Z[i] into the ring of integers Z, N : Z[i] → Z.

Finite fields again

We won't find any examples of finite integral domains that aren't fields because there aren't any.

Theorem: If R is a finite integral domain, then R is a field. Proof. Let x be a nonzero element of R. Consider the positive powers of x: x, x^2, x^3, ..., x^n . . . .

Since there are infinitely many powers, but only finitely many elements in R, therefore at least two distinct powers are equal. Let, then, x m = x n with m < n. Cancel x m from each side of the equation to conclude x n−m = 1. Therefore, the reciprocal of x is x n−m−1. Therefore, every nonzero element has an inverse.

This theorem can be used to give a short proof that Zp is a field when p is a prime, since it's easy to show that Zp is an integral domain. We'll show it has no zero-divisors. Suppose that xy ≡ 0 (mod p). Then p|xy. But if a prime divides a product, it divides one of the factors, so either p| x or p| y, in other words, either x ≡ 0 (mod p) or y ≡ 0 (mod p). Thus, Zp is an integral domain, and hence, by the above theorem, it's a field. Our earlier, more complicated proof used the extended Euclidean algorithm to find an inverse for x. That's actually a much more efficient way to find the inverse than to look through the powers of x.

CONCLUSION:

In this paper we found that one important instance of an integral domain is that of the Gaussian integers Z[i]. Its elements are of the form x + yi where x, y b £ Z, so they can be viewed as a lattice of points in the complex plane. We can make sure that Z[i] is closed under addition, subtraction, multiplication, and includes 1, so it is a subring of the field C. Therefore, it's an integral domain.

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