"On the Convergence of Aluthge Sequence: A Review"

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Abstract – For $0 < \lambda < 1$, the λ -Aluthge sequence $\{\Delta^m_\lambda(X)\}_{m\in N}$ converges if the nonzero eigenvalues of $X \in C^{n\times n}$ have distinct moduli, where $\Delta_\lambda(X) := P^\lambda UP^{1-\lambda}$ if $X = UP$ is a polar decomposition of X.

INTRODUCTION

Given $X \in C^{n\times n}$, the polar decomposition [9] asserts that $X = UP$, where U is unitary and P is positive semi definite, and the decomposition is unique if X is nonsingular[20]. Though the polar decomposition may not be unique, the Althuge transform [1] of X: $\Delta(X) := P^{1/2}UP^{1/2}$ ($P^{1/2}XP^{-1/2}$ if X is nonsingular) is well defined [17, Lemma 2]. Aluthge transform has been studied extensively, for example, [1, 2, 3, 4, 5, 7, 8, 11, 12, 13, 14, 16, 17]. Recently Yamazaki [16] established the following interesting result (1.1) $\lim_{m\to\infty} \|\Delta^m(X)\| = r(X)$, where r(X) is the spectral radius of X and $\|X\| := \max_{\|v\|_2=1} \|Xv\|_2$ is the spectral norm of X. Suppose that the singular values $s_1(X), \ldots, s_n(X)$ and the eigenvalues $\lambda_1(X), \ldots, \lambda_n(X)$ of X are arranged in nonincreasing order $s_1(X) \geq s_2(X) \geq \cdots \geq s_n(X)$, $|\lambda_1(X)| \geq |\lambda_2(X)| \geq \cdots \geq |\lambda_n(X)|$. Since $\|X\| = s_1(X)$ and $r(X) := |\lambda_1(X)|$, the following result of Ando [3] is an extension of (1.1). Theorem 1.1. (Yamazaki-Ando) Let $X \in C^{n\times n}$. Then (1.2) $\lim_{m\to\infty} s_i(\Delta^m(X)) = |\lambda_i(X)|$, $i = 1, \ldots, n$[20].

REVIEW OF LITERATURE;

Aluthge transform $\Delta(T)$ is also defined for Hilbert space bounded linear operator T [17] and (1.1) remains true [16]. Yamazaki's result (1.1) provides support for the following conjecture of Jung et al [11, Conjecture 1.11] for any $T \in B(H)$ where B(H) denotes the algebra of bounded linear operators on the Hilbert space H. Conjecture 1.2. Let $T \in B(H)$. The Aluthge sequence $\{\Delta^m(T)\}_{m\in N}$ is norm convergent to a quasinormal $Q \in B(H)$, that is, $\|\Delta^m(T) - Q\| \to 0$ as $m \to \infty$, where $\|\cdot\|$ is the spectral norm. It is known [11, Propositioin 1.10] that if the Aluthge sequence of $T \in B(H)$ converges, its limit L is quasinormal, that is, L commutes with L^*L, or equivalently, $UP = PU$ where $L = UP$ is a polar decomposition of L [9]. However very recently it is known [7] that Conjecture 1.2 is not true for infinite dimensional Hilbert space. Ch¯o, Jung and Lee [7, Corollary 3.3] constructed a unilateral weighted shift operator $T : \ell^2(N) \to \ell^2(N)$ such that the sequence $\{\Delta^m(T)\}_{m\in N}$ does not converge in weak operator topology[20]. They also constructed [7, Example 3.5] a hyponormal bilateral weighted shift $B : \ell^2(Z) \to \ell^2(Z)$ such that $\{\Delta^m(B)\}_{m\in N}$ converges in the strong operator topology, that is, for some $L : \ell^2(Z) \to \ell^2(Z)$, $\|\Delta^m(B)x-Lx\| \to 0$ as $m \to \infty$ for all $x \in \ell^2(Z)$, where $\|x\|$ is the norm induced by the inner product. However $\{\Delta^m(B)\}_{m\in N}$ does not converge in the norm topology. So the study of Conjecture 1.2 is reduced to the finite dimensional case $C^{n\times n}$. Since the three (weak, strong, norm) topologies coincide and quasinormal and normal coincide [9] in the finite dimensional case, the limit points of the Aluthge sequence are normal [13, Proposition 3.1], [3, Theorem 1]. Also see [11, Proposition 1.14]. Moreover the eigenvalues of $\Delta(X)$ and the eigenvalues of X are identical, counting multiplicities[20].

CONVERGENCE OF ITERATED ALUTHGE TRANSFORM SEQUENCE FOR DIAGONALIZABLE MATRICES:

The iterates of usual Aluthge transform $\Delta^n_{1/2}(T)$ converge to a normal matrix $\Delta^\infty_{1/2}(T)$ for every diagonalizable matrix $T \in M_r(C)$ (of any size). We also proved in [21] the smoothness of the map $T \mapsto \Delta^\infty_{1/2}(T)$ when it is restricted to a similarity orbit, or to the (open and dense) set $D^*_r(C)$ of invertible r×r matrices with r different eigenvalues. The key idea was to use a dynamical systems approach to the Aluthge transform, thought as acting on the similarity orbit of a diagonal invertible matrix. Recently, Huajun Huang and Tin-Yau Tam [22] showed, with other approach, that the iterates of every λ-Aluthge transform Δ^n

λ (T) converge, for every matrix T ∈ Mr(C) with all its eigenvalues of different moduli.

CONCLUSION:

In this paper, we study the general case of λ-Aluthge transforms by means of a dynamical systems approach. This allows us to generalize Huajun Huang and Tin-Yau Tam result for every diagonalizable matrix T ∈ Mr(C), as well as to show regularity results for the two parameter map (λ, T) 7→ ∆∞ λ (T) = limn∈N ∆n λ (T).

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