An Analysis upon Some Remarks on Gaussian Isoperimetric Inequality: A New Approach

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Abstract – The Gaussian isoperimetric inequality, and its related concentration phenomenon, is one of the most important properties of Gaussian measures. These notes aim to present, in a concise and self-contained form, the fundamental results on Gaussian processes and measures based on the isoperimetric tool. In particular, our exposition will include, from this modern point of view, some of the by now classical aspects such as integrability and tail behavior of Gaussian semi norms, large deviations or regularity of Gaussian sample paths. We will also concentrate on some of the more recent aspects of the theory which deal with small ball probabilities.

We give a martingale proof of Gaussian isoperimetry, which also contains Bobkov's inequality on the two-point space and its extension to non-symmetric Bernoulli measures. We derive the equivalence of different forms of Gaussian type isoperimetry. This allows us to prove a sharp form of Bobkov's inequality for the sphere and to get new isoperimetric estimates for the unit cube.

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INTRODUCTION

A popular isoperimetric inequality is known as the classical isoperimetrical inequality. It was proposed by Zenodorus, a Greek mathematician. This document exposes the applications of isoperimetric inequalities in modern fields. An obvious application of isoperimetric inequalities is in area optimization. This case was exploited by Queen Dido of Carthage. We will not describe Queen Dido's application of the inequality, interested readers might find [8] informative. This singular application, area optimization, seems insignificant when one considers the complexity of proving the classical isoperimetric inequality. This document will provide applications which are specific to computers and computer networks as they have become ubiquitous tools in the 21st century.

In computer networks, data is transferred from sender to receiver as a sequence of on and off signals via a communication channel. The issue with data transfer is that it is sometimes done through unreliable or noisy channels hence loss of data is inevitable. This poses the challenge of how to determine if data contains errors at the receiver's side. Information theory, a subject mostly attributed to Claude E. Shannon, presents concepts such as error correction and detection which allow the detection of error and restoration of data.

In this study, we extend a Brownian approach to (1.2) due to Capitaine, Hsu and Ledoux. We get a unified proof of (1.3) and (1.2), and an extension of (1.3) to an isoperimetric inequality for non-symmetric Bernoulli measures. This section contains a proof of the equivalence of different forms of isoperimetry on the Gaussian model. It follows from works by Wang and by Bakry and Ledoux that for any probability measure $d\mu(x) = e^{-V(x)}\,dx$ on $\mathbb{R}^n$, with $V'' \geq \alpha\, Id_{\mathbb{R}^n}$ for some $\alpha \in \mathbb{R}$, and such that $\iint \exp(\varepsilon\,|x-y|^2)\,d\mu(x)d\mu(y) < \infty$ for some $\varepsilon > \sup(0, -\alpha)$, there exists $c > 0$ such that for every Borel set $A \subset \mathbb{R}^n$

$$\mu^+(A) \geq c\, I(\mu(A)). \qquad (1)$$

A simple proof of this fact for log-concave probability measures is given by Bobkov:(1.4) is equivalent to the existence of a number $\varepsilon > 0$ such that $\int \exp(\varepsilon|x|^2)\,d\mu(x) < \infty$ (Herbst condition). Moreover he proves that (4) implies that for all locally Lipschitz functions $f : \mathbb{R}^n \to [0,1]$,

$$I\left(\int_{\mathbb{R}^n} f\,d\mu\right) \leq \int_{\mathbb{R}^n} \left(I(f) + \frac{1}{c}|\nabla f|\right) d\mu. \quad (2)$$

The constants provided by these results are not very good. Sharp ones are given by Bakry and Ledoux: under the hypothesis $V'' \geq c^2 Id_{\mathbb{R}^n}$, one has for $d\mu(x) = e^{-V(x)}\,dx$ and every /as above

$$I\left(\int_{\mathbb{R}^n} f\,d\mu\right) \leq \int_{\mathbb{R}^n} \sqrt{I^2(f) + \frac{1}{c^2}|\nabla f|^2}\, d\mu. \quad (3)$$

Notice that the case $\mu = \gamma_n$ and c = 1 gives).

It is clear that (3) implies (2), which implies (1). We will show that they are equivalent, with the same constant c. The proof strongly relies on the Gaussian model. Then, we give a sharp form of Bobkov's inequality for spheres, using the Gaussian isoperimetric function *I.* Finally, we improve the isoperimetric estimates of Hadwiger for the unit cube in $\mathbb{R}^n$. In particular, we recover the following result of Hadwiger: among subsets of measure 1/2 of the unit cube, half-cubes have the smallest boundary measure.

GAUSSIAN NOISE STABILITY AND GAUSSIAN ISOPERIMETRIC INEQUALITY

One of the oldest mathematical problems is the isoperimetric inequality in two dimensions. An isoperimetric inequality connects the volume of a set with its surface area. The isoperimetric inequality in $\mathbb{R}^n$ asserts that for every compact subset $A \subset \mathbb{R}^n$ with smooth boundary ∂A and every ball $B \subset \mathbb{R}^n$ with $\mathrm{vol}_n(A) = \mathrm{vol}_n(B)$ we have the inequality $\mathrm{vol}_{n-1}(\partial A) \geq \mathrm{vol}_{n-1}(\partial B).$

An equivalent formulation is given by

$$\mathrm{vol}_n(A_r) \geq \mathrm{vol}_n(B_r)$$

Where $M_r := \{x \in X; d(x,y) \leq r$ for some $y \in M\}$ is the r-extension of a set $M \subset X$ in a metric space *(X,d).*

(Clearly, we consider $X = \mathbb{R}^n$ with the Euclidean distance.)
The equivalence of these two statements can be proved via Minkowski's formula

$$\mathrm{vol}_{n-1}(\partial A) = \liminf_{r \downarrow 0} \frac{1}{r}\left[\mathrm{vol}_n(A_r) - \mathrm{vol}_n(A)\right] \quad (4)$$

for a sufficiently smooth boundary ∂A

Moreover, there is the following isoperimetric inequality on the sphere S_ρ^N in $\mathbb{R}^{N+1}$ with radius ρ. The sphere S_ρ^N is equipped with the geodesic distance as metric and the normalized rotationally invariant measure σ_ρ^N

Theorem 1. *Let $A \subset S_\rho^N$ be a measurable subset and $B \subset S_\rho^N$ a geodesic ball such that $\sigma_\rho^N(A) \geq \sigma_\rho^N(B)$ then $\sigma_\rho^N(A_r) \geq \sigma_\rho^N(B_r)$ for every $r \geq 0$*

We denote the standard Gaussian measure on $\mathbb{R}^n$ by γ, the one-dimensional standard Gaussian measure by γ^1 and the cummulative distribution function of γ^1 by

$$\Phi(s) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-t^2/2} \mathrm{d}t.$$

The Gaussian measure γ can be considered as the limit of $\sigma_{\sqrt{N}}^N$ for $N \to \infty$ in the following sense: If $\pi_{N+1,n}: \mathbb{R}^{N+1} \to \mathbb{R}^n$ denotes the projection onto the first n-components of a vector in $\mathbb{R}^{N+1}\ (N \geq n)$ then we have $\lim_{N\to\infty} \sigma_{\sqrt{N}}^N\left(\pi_{N+1,n}^{-1}(A) \cap S_{\sqrt{N}}^N\right) = \gamma(A)$ for all measurable $A \subset \mathbb{R}^n$. A proof of this fact can be found in.

As geodesic balls on S_ρ^N arise as the intersection of the sphere with half-spaces it is not difficult to believe that in the Gaussian isoperimetric inequality, half-spaces will fill the role of balls and geodesic balls in $\mathbb{R}^n$ and on S_ρ^N, respectively. A half-space *H* in $\mathbb{R}^n$ is a set of the form $H = \{x \in \mathbb{R}^n; \langle x, u\rangle \geq a\}$ for some $a \in \mathbb{R}$ and a unit vector $u \in \mathbb{R}^n$

Therefore, the following result can be seen as an infinite dimensional version of Theorem.

Theorem 2. *If $A \subset \mathbb{R}^n$ is measurable and H is a half-space with $\gamma(A) \geq \gamma(H)$ then $\gamma(A_r) \geq \gamma(H_r)$ for ever $r \geq 0$* Since $\gamma(\{x \in \mathbb{R}^n; \langle x, u\rangle \geq a\}) = \Phi(a)$ the theorem can be stated equivalently as

$$\gamma(A_r) \geq \Phi(\Phi^{-1}(\gamma(A)) + r) \quad (5)$$

for every $r \geq 0$.

Using Minkowski's formula (4) as a motivation to define the Gaussian surface area of a measurable set $A \subset \mathbb{R}^n$ via

$$\gamma^+(A) := \liminf_{r \downarrow 0} \frac{\gamma(A_r) - \gamma(A)}{r}$$

where $A_r := \{x \in \mathbb{R}^n; |x-y| \le r$ for some $y \in A\}$ is the r-extension of *A* we can state the Gaussian isoperimetric inequality in the following form.

Theorem 3 (Gaussian isoperimetric inequality). *If* $A \subset \mathbb{R}^n$ *is a measurable subset then* $\gamma^+(A) \ge \gamma^1(\Phi^{-1}(\gamma(A))) = I(\gamma(A))$.

Here, we used the notation $I(x) := \gamma^1(\Phi^{-1}(x))$. In this section we will deduce this inequality from a generalization first proved by Borell.

For /r-dimensional standard Gaussian vectors *X* and *Y* with $\mathbb{E}X_iY_j = \rho\delta_{ij}$ and $0 < \rho < 1$ he introduced the Gaussian noise stability $\Pr_\rho(X \in A, Y \in A)$, i.e. the probability that *X* and *Y* lie both in a measurable set $A \subset \mathbb{R}^n$ and showed that it fulfills

$$\Pr_\rho(X \in A, Y \in A) \le \Pr_\rho(X_1 \le a, Y_1 \le a) \tag{6}$$

For $0 < \rho < 1$ and $a := \Phi^{-1}(\gamma(A))$. Note that *a* is chosen such that $\gamma(A) = \gamma(\{x \in \mathbb{R}^n; x_1 \le a\})$. This means that half-spaces maximize the Gaussian noise stability among all measurable sets with the same Gaussian measure. In this section we will see that Theorem 3 follows in the limit $\rho \to 1$ indeed from this inequality or more precisely from the Gaussian noise sensitivity, i.e. the inequality

$$\Pr_\rho(X \in A, Y \notin A) \ge \Pr_\rho(X_1 \le a, Y_1 \ge a) \tag{7}$$

which is equivalent to (6). The proof follows a semigroup argument proposed by Ledoux.

In this section we will verify that $\Pr_\rho(X \in A, Y \in A) = \mathcal{S}_{\rho^2}(A)$ for every measurable subset $A \subset \mathbb{R}^n$ where

$$\mathcal{S}_\rho(A) = \mathbb{P}\left(\sqrt{\rho}X + \sqrt{1-\rho}Y \in A, \sqrt{\rho}X + \sqrt{1-\rho}Y' \in A\right) \tag{8}$$

For $0 \le \rho \le 1$ where *X,Y* and *Y′* are independent standard Gaussian vectors in $\mathbb{R}^n$. In the following we will refer to this quantity as the *Gaussian noise stability.* With this definition, (6) can be formulated as in the following theorem.

Theorem 4 (Gaussian noise stability). *If* $A, H \subset \mathbb{R}^n$ *are measurable subsets such that H is a half-space with* $\gamma(A) = \gamma(H)$ *and* $0 \le \rho \le 1$ *then* $\mathcal{S}_\rho(A) \le \mathcal{S}_\rho(H)$

Definition gives rise to the following generalization. For a measurable subset $A \subset \mathbb{R}^n$ we define the r-stability of *A* for *r* > 1 and $0 < \rho < 1$ through $\mathcal{S}_\rho^r(A) := \mathbb{E}\left[\mathbb{P}\left(\sqrt{\rho}X + \sqrt{1-\rho}Y \in A \middle| X\right)^r\right]$

We show in this section that this quantity introduced by E. Mossel is a generalization of the Gaussian noise stability. Indeed, we will prove that $\mathcal{S}_\rho(A) = \mathcal{S}_\rho^2(A)$ for every measurable subset $A \subset \mathbb{R}^n$ and $0 < \rho < 1$ Therefore, Theorem 2 and thus (6) will follow from the next result which is the main result of these notes.

In order to describe the equality cases in the theorem properly, we need the following notation. For a measurable subset $B \subset \mathbb{R}^n$ we define its center of mass with respect to the Gaussian measure $v(B) = (v_i(B))_{i=1}^n$ with the components

$$v_i(B) = \int_B x_i \gamma(x) \mathrm{d}x.$$

If $v_i(B) = 0$ for $i = 1, \ldots, n$ then we set $v(B) = e_1$. Moreover, we $q(B) := \|v(B)\|_2$ set, i.e. the distance of this center of mass from the origin. This enables us to associate the half-space

$$H(B) := \{x \in \mathbb{R}^n; \langle v(B), x\rangle \ge \alpha\} \tag{9}$$

to *B* where *a* is chosen such that $\gamma(B) = \gamma(H(B))$. The importance of *H(B)* is caused by the fact that the symmetric difference of *B* and *H(B*) is a null set if $\mathcal{S}_\rho^r(B)$ agrees with the r-stability of a half-space with Gaussian measure $\gamma(B)$. More precisely, the following generalization of Theorem 1.4 holds true.

Theorem 5. *For a measurable subset* $A \subset \mathbb{R}^n$, $0 < \rho < 1$ *and* $r > 1$ *we have*

$$\mathcal{S}_\rho^r(H(A)) \ge \mathcal{S}_\rho^r(A) \tag{10}$$

Equality holds if and only if the symmetric difference of A and H(A) has measure zero.

We conclude this section with an outline of the remaining contents of this note. This section contains some notations and remarks as well as an application of the Gaussian isoperimetric inequality to concentration of the Gaussian measure. In this section, we give a proof of the main result of this note, Theorem [5], based on techniques from stochastic calculus. The following section consists of a proof of the Gaussian isoperimetric inequality which uses Theorem [5]. In the end of this work, we collect some abstract auxiliary results employed in this section.

QUANTITATIVE ISOPERIMETRIC INEQUALITY ON THE SPHERE

Recent years have seen an increasing interest in quantitative isoperimetric inequalities motivated by classical papers by Bernstein and Bonnesen. In the Euclidean case the optimal result states that if *E* is a set of finite measure in W^1 then $\delta(E) \geq c(n)\alpha(E)^2$ holds true. Here the *Fraenkel asymmetry index* is defined as $\alpha(E) := \min \frac{|E \Delta B|}{|B|}$, where the minimum is taken among all balls $B \subset \mathbb{R}^n$ with $|B| = |E|$, and the isoperimetric gap is given by $\delta(E) := \frac{P(E) - P(B)}{P(B)}$.

The stability estimate has been generalized in several directions, for example to the case of the Gaussian isoperimetric inequality, to the Almgren higher co dimension isoperimetric inequality and to several other isoperimetric problems. In the present study we address the stability of the isoperimetric inequality on the sphere by Schmidt stating that if $E \subset S^n$ is a measurable set having the same measure as a geodesic ball $B_\vartheta \subset S^n$ for some radius $\vartheta \in (0, \pi)$, then $\mathbf{P}(E) \geq \mathbf{P}(B_\vartheta)$, with equality if and only if E is a geodesic ball. Here, P(.E) stands for the perimeter of E, that is $\mathbf{P}(E) = \mathcal{H}^{n-1}(\partial E)$ if E is smooth and $n \geq 2$ throughout the whole study.

In view of the previously mentioned stability results the natural counterpart of (1.1) would be the inequality

$$\frac{\mathbf{P}(E) - \mathbf{P}(B_\vartheta)}{\mathbf{P}(B_\vartheta)} \geq c(n)\alpha(E)^2 \tag{11}$$

where now the Fraenkel asymmetry index is defined by

$$\alpha(E) := \min \frac{|E \Delta B_\vartheta|}{|B_\vartheta|} \tag{12}$$

The minimum is taken over all geodesic balls $B_\vartheta \subset S^n$ with $|E| = |B_\vartheta|$. Notice, that we are denoting the $\mathcal{H}^n$-measure of a set E by $|E|$. When compared with inequality (11), even if it looks similar, has a completely different nature; in fact is scaling invariant (i.e. invariant under homotheties), while there is no scaling at all on S^n. Indeed, it would be quite easy to adapt one of the different arguments in the study in order to prove (11) with a constant depending additionally on the volume of the set E, but blowing up as $\vartheta \downarrow 0$. In fact, the difficult case is when the set E has a small volume sparsely distributed over the sphere. In this situation a localization argument aimed to reduce the problem to the flat Euclidean estimate cannot work.

To state our main result we introduce the oscillation index $\beta(E)$ of a set $E \subset S^n$

$$\beta^2(E) := \tfrac{1}{2} \min_{p_o \in S^n} \int_{\partial E} \left|\nu_E(x) - \nu_{B_{\vartheta(x)}(p_o)}(x)\right|^2 d\mathcal{H}^{n-1}(x) \tag{13}$$

Where $\nu_E(x)$ is the outer unit normal to Eat the point $x \in \partial E$ (contained in the tangent plane to S^n at x% i.e. $\nu_E(x) \cdot x = 0$) and $\nu_{B_{\vartheta(x)}(p_o)}(x)$ is the outer unit normal to the geodesic ball $B_{\vartheta(x)}(p_o)$ centered at p_o whose boundary passes through x. Note, that the Fraenkel asymmetry $\alpha(E)$ measures the L^1-distance between the set E and the optimal geodesic ball of the same volume, while the oscillation index $\beta(E)$ is a measure for the distance between the distributional derivatives of χ_E and of χ_{B_ϑ}, where B_ϑ is an optimal geodesic ball in (13) of the same volume as E. Alternatively, the oscillation index can be viewed as an excess functional between ∂E and ∂B_ϑ. The two indices are related by a Poincare-type inequality, stating that

$$\beta^2(E) \geq c(n)\mathbf{P}(B_\vartheta)\alpha^2(E) \tag{14}$$

The main result of the present study can now be formulated as follows:

Theorem 6. There exists a constant c(n) such that for any set $E \subset S^n$ of finite perimeter with volume for some $\vartheta \in (0, \pi)$, the following inequality holds

$$\mathbf{D}(E) := \mathbf{P}(E) - \mathbf{P}(B_\vartheta) \geq c(n)\beta^2(E) \tag{14}$$

We mention that (14) is the counterpart for the sphere of a similar inequality, where a suitable definition of oscillation index in the euclidean case was introduced for the first time. Note that as in the euclidean case by combining (1.23) with (1.22) immediately yields the stability inequality (11). On the other hand it is clear that (14) is stronger than (11), $\mathbf{P}(E) - \mathbf{P}(B_\vartheta) \leq \beta^2(E)$ the starting point for the proof of Theorem 1.9 is a Fuglede-type stability result aimed to establish (14) in the special case of sets $E \subset S^n$ whose boundary can be written as a radial graph over the boundary of a ball $B_\vartheta(p_o)$ with the same volume.

To establish such a result one could follow in principle the strategy used in the Euclidean case. However, to deduce (14) for radial graphs with a constant not depending on the volume needs much more care in the estimations. The main difficulty arises when passing from the special situation of radial graphs to arbitrary sets. To deal with this issue we need to change significantly the strategies developed.

To explain where the major difficulties come from, we observe that the oscillation index can be re-written in the form

$$\beta^2(E) = \mathbf{P}(E) - (n-1) \max_{p_o \in S^n} \int_E \frac{x \cdot p_o}{\sqrt{1-(x \cdot p_o)^2}} \, d\mathcal{H}^{n-1}.$$

From this formula it is clear that the core of the proof is to provide estimates for the singular integral

$$\int_E \frac{x \cdot p_o}{\sqrt{1-(x \cdot p_o)^2}} \, d\mathcal{H}^{n-1} \tag{15}$$

and its maximum with respect to p_0, independent of the volume of E. This requires new technically involved ideas and strategies. In fact, in the contradiction argument used to deduce (14) for general sets from the case of a radial graph we need to show that all the constants are independent of the volume of E. The arguments become particularly delicate when the volume of E is small. In this case inequality (14) shows a completely different nature depending on the size of the ratio $\beta^2(E)/\mathbf{P}(B_\vartheta)$. In fact, if $|E| \to 0$ and also $\beta^2(E)/\mathbf{P}(B_\vartheta) \to 0$, then E behaves asymptotically like a flat set, i.e. a set in $\mathbb{R}^n$ and inequality (14) can be proven by reducing to the euclidean case, rescaling and then arguing as when E has large volume. However, the most difficult situation to deal with is when $|E| \to 0$ and $\beta^2(E)/\mathbf{P}(B_\vartheta) \to \eta_o > 0.$

This case has to be treated with ad hoc estimates for the singular integral (15).

BROWNIAN PROOF OF BOBKOV'S INEQUALITIES

In order to simplify the notation we work with real-valued processes and functions. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion on $\mathbb{R}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$ and sucli that $B_0 = 0$. We assume that all the processes appearing below are adapted with respect to this filtration.

Proposition 1 Let $(M_t)_{t\geq 0}$ and $(N_t)_{t\geq 0}$ be $(\mathcal{F}_t)_{t\geq 0}$ real-valued martingales with $M_t = M_0 + \int_0^t m_s \, dB_s$, $N_t = N_0 + \int_0^t n_s \, dB_s$. And let $A_t = A_0 + \int_0^t a_s \, ds$ be an increasing process, such that A_t is bounded for every $t \geq 0$ *and* $A_0 \geq 0.$ Assume that $a_t |N_t|^2 \geq |m_t|^2$ for every $t \geq 0$. And that for some $\varepsilon \in (0, 1/2)$, we have $M_t \in [\varepsilon, 1-\varepsilon]$ for every $t \geq 0$. Then $\left(\sqrt{I^2(M_t) + A_t |N_t|^2}\right)_{t\geq 0}$ is a sabmartingale.

The result remains true, with essentially the same proof, when M_t is a real martingale and N_t a vector-valued martingale with respect to the *n*- dimensional Brownian motion $(B_t^{(n)})$. In this case. (m_t) is a vector process. $dM_t = m_t \cdot dB_t^{(n)}$ is the scalar product in $\mathbb{R}^n$ and (n_t) is a matrix-valued process. The condition above remains $a_t |N_t|^2 \geq |m_t|^2$. This time with the Euclidean norm.

Proof: Let $J: \mathbb{R} \to \mathbb{R}$ be a positive C^1 function, constant outside [0.1] and such, that $J(x) = I(x)$ when $\varepsilon \leq x \leq 1-\varepsilon$. Let $F(x,y,t) = \sqrt{J^2(x) + ty^2}$ Direct computations give

$$\begin{aligned} \frac{\partial^2 F}{\partial x^2} &= \frac{1}{F^3}\left(tJ'^2(x)y^2 + J^3(x)J''(x) + ty^2 J(x)J''(x)\right) \\ \frac{\partial^2 F}{\partial y^2} &= \frac{tJ^2(x)}{F^3}; \quad \frac{\partial^2 F}{\partial x \partial y} = -\frac{tyJ(x)J'(x)}{F^3}. \end{aligned}$$

Writing Q_t for the triple (M_t, N_t, A_t), we get by Itô's formula

$$X_t := F(Q_t) = F(Q_0) + \int_0^t \left(\frac{\partial F}{\partial x}(Q)\, dM + \frac{\partial F}{\partial y}(Q)\, dN\right) + \int_0^t \Delta(s)\, ds,$$

with

$$\Delta(s) = \frac{\partial F}{\partial t} a_s + \frac{1}{2}\left(\frac{\partial^2 F}{\partial x^2}(Q_s)\, m_s^2 + 2\frac{\partial^2 F}{\partial x \partial y}(Q_s)\, m_s n_s + \frac{\partial^2 F}{\partial y^2}(Q_s)\, n_s^2\right)$$

Since the stochastic integral has a bounded integrand, it is a martingale. Hence (X_t) is a martingale plus $\int_0^t \Delta(s)\, ds$. But since $J(M_t)$ and $I(M_t)$ coincide, we get using the relation $I\,I'' = -1$ and omitting the variables

$$2F^3\Delta = F^2N^2a + (I'^2AN^2 - I^2 - AN^2)m^2 - 2(II'AN)mn + I^2An^2.$$

Since $N^2a \geq m^2$, we have $F^2N^2a \geq (I^2 + AN^2)m^2$, hence

$$2F^3\Delta \geq A(I'^2N^2m^2 - 2II'Nmn + I^2n^2) = A(I'Nm - In)^2 \geq 0,$$

thus (X_t) is a sub martingale.

The preceding computation is not difficult. but does not really explain why the result was intuitively clear. Given a non-negative semimartingale *Z* such that $Z + dZ = Z + q\,dB + r\,dt$. Itô's formula shows that $\sqrt{Z}$ is a submartingale precisely when the formal second degree polynomial in the β variable $T = Z + q\beta + r\beta^2$ has a non-positive discriminant $q^2 - 4Zr \leq 0$, or in other words when $T \geq 0$ for every real value of β. When $Z = I^2(M) + AN^2$, our formal expression is equal to

$$T = (I^2 + 2II'm\beta + I'^2m^2\beta^2) + II''m^2\beta^2 + aN^2\beta^2 + A(N^2 + 2Nn\beta + n^2\beta^2) \geq$$
$$\geq (I + I'm\beta)^2 + A(N + n\beta)^2 \geq 0$$

since $II''m^2 + aN^2 = -m^2 + aN^2 \geq 0$. The trick is simply that the increase of A_t (multiplied by N_t^2) must compensate the fact that $II'' < 0$. If we were trying to do the same for a different function *J.* we see that all is needed is that $(JJ'')(M_t)\, m_t^2 + a_t N_t^2 \geq 0$ for every *t.*

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