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A Pairing on the Group of Degree Zero Divisors of A Curve Over A Number Field

Shweta

Research Scholar, Singhania University, Rajasthan, India

ABSTRACT: ***In this paper we construct a pairing on the group of degree zero divisors of a curve over a number field. This is accomplished by passing from divisors of the curve to divisors of an associated scheme and then employing an Arakelov intersection theory. Arakelov geometry studies a scheme X over the ring of integers Z, by putting Hermitian metrics on holomorphic vector bundles over X(C), the complex points of X. This extra Hermitian structure is applied as a substitute, for the failure of the scheme Spec(Z) to be a complete variety..***

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DISCUSSION

To carry this out we need to define volume forms on the Riemann surface associated to the curve. This is the main topic of the first section. We must also have some understanding of how the presence of singular fibers in the scheme can affect things. This is done through the study of Picard functors. It is the bulk of section two. Finally in the third section of this chapter we define the pairing and study its behaviour under change of base field.

Let K be a number field and 0 its rings of integers. Let C be a regular curve over 0 such that g, the genus of $C \otimes K,$ is strictly greater than zero. Let p be a maximal ideal of 0 and let3be the set of prime Weil divisors of C which lie over p. Let x_p^i be the generic point of C_p^i and let O_p^i be the local ring of C at $x_p^i.$ Let $n_p^i = \text{length}(O_p^i)$ So n_p^i is the multiplicity of $C \otimes k(p)$ along $C_p^i.$ Let n_p be the least common divisor of the set $\{n_p^i\}_i$. For the rest of the chapter we assume for all maximal ideals p. $n_p = 1$

REMARKS:

1. If X is a smooth curve over K of genus greater than zero and possessing a K-rational point, then we may construct a regular curve V over 0 which satisfies III.2 with the property that $X \simeq V \otimes K.$

2. Let C' be obtained by blowing up the scheme C at a closed point. Then C* will be a regular curve over 0 and, by [Re), *C'* will also satisfy II.3. Let $\phi_i: K \to C$ be an embedding. Let $C_i = C \otimes_{\phi_i} \mathbb{C}.$

We consider C_i on a smooth connected compact Riemann surface. Let $\omega_i^1, \ldots, \omega_i^g$ be a basis for the holomorphic differentials of C_i, normalized so that

$$\frac{\sqrt{-1}}{2\pi} \int_{C_i} \omega_i^j \wedge \overline{\omega}_i^k = \delta_{jk}.$$

Let

$$d\mu_i = \frac{\sqrt{-1}}{2\pi g} \sum_j \omega_i^j \wedge \overline{\omega}_i^j.$$

Clearly $d\mu_i$ is a volume form on C_i Let [,] denote the pairing on Arakelov divisors of C which results from the use of the forms $d\mu_i$ above.

Let $\text{Div}^o C$ denote the group of Weil divisors of C which have degree zero when restricted to $C \otimes K.$ Let $\text{Pic}^o C$

denote the quotient of Div^o by the subgroup of principal Weil divisors of C. The pairing [,] induces a pairing on $Pic^o C$, which we also denote by [,]. Let D be a divisor of $C \otimes K$ which has degree zero. Let $\overline{D}$ be the divisor of C obtained from D by scheme-theoretic closure. If D is principal, $\overline{D}$ may not be principal. In general $\overline{D}$ will differ from a principal divisor of C by some divisor F. This divisor F will lie over the primes p of 0 such that $C \otimes k(p)$ has more than one component.

Therefore, to use the pairing [,] to construct a pairing on $Pic^O C \otimes K$, we must augment the map $D \to \overline{D}$.

Let p be a maximal ideal of 0 and let $\{C_p^i\}$ be the set of prime divisors of C which lie over p. Let D be a degree zero divisor of $C \otimes K$ and let $\overline{D}$ denote the divisor of C obtained from D by scheme-theoretic closure.

Proposition Is There exists integers m_p and d_p^i such that

$$[m_p\overline{D} + \sum_i d_p^i C_p^i, E] = 0$$

for all divisors E of C which lie over p. Furthermore the integer m_p depends only on p and C, not on D.

Proof: Let o_p be the localization of 0 at p, and let k(p) be the residue field of o_p. Let R be a discrete valuation ring such that:

i) R is an o_p module

ii) $C \otimes R$ is regular.

Let q_R be the maximal ideal of R, L_R the fraction field of R and π_R the projection from $C \otimes R$ to $C \otimes O_p$.

Let $\{C_R^j\}$ be the set of prime divisors of $C \otimes R$ which lie over q_R. Let (,) denote the Lichtenbaum-Shaferavitcl pairing on $C \otimes R$. We will say that (*) holds for R if the follwoing statement is true:

There is a positive integer m_R such that: Given a divisor D of $C \otimes L_R$ which has degree zero, there (*) are integers $\{d_R^j\}$ such that:

$$(m_R\overline{D} + \sum_j d_R^j C_R^j, E) = 0,$$

for all divisors E of $C \otimes R$ which lie over q_R. To prove the proposition we must show that (*) holds for the ring $R = O_p$. Let $\hat{O}$ be the completion of O_P. To show that (8) holds for O_p it is sufficient to show that (*) holds for the ring $R = \hat{O}$. Let F_n denote the finite field extension of k(p), of degree n. Let S_n be the discrete valuation ring which is an unramified extension of $\hat{O}$ and whose residue field is F_n, S_n exists by Hensel's lemma, see EGA_{IV}. Let S be the direct limit of the S_n's, S is a discrete valuation ring whose residue field is $\overline{k(p)}$, the algebraic closure of k(p). The ring S is also known as the strict henselization of $\hat{O}$. The rings S_n satisfy conditions i) and ii) above, hence so does S.

We show below, in Proposition 2, that (*) holds for the ring S. To finish the proof of the proposition we must show that this implies that (*) holds for the ring $\hat{O}$. This is accomplished in two steps: First we show that if (*) holds for S, then (*) holds for some intermediate ring S_N; then we show that if (*) holds for some S_N, it also holds for $\hat{O}$.

Let $\{C_p^i\}$ be the set of prime divisors of $C \otimes \hat{O}$ which lie over p, these divisors correspond to the prime divisors of C which lie over p. Each C_p^i is an integral scheme over k(p), but these schemes may not be geometrically integral. Let k_i be the algebraic closure of k(p) in the field of rational functions of C_p^i. Let n_i be the degree of k_i over k(p) and let N be the maximum of the n_i's. F_N, S_N as above. Let q_N be the maximal ideal of S.

Let π_1 be the projection map $C \otimes S \to C \otimes S_N$. If E is a prime divisor of $C \otimes S_N$ which lies over q_N, then E is geometrically integral, therefore the pullback of E to $C \otimes S$ has support along a prime divisor:

$$\pi_1^* E = nE',$$

E' a prime divisor of $C \otimes S$. Since S is unramified over S_N, we have n » 1. Let F_1, F_2 be two divisors of $C \otimes S_N$, with one lying over q_N. By the corollary to Proposition 11 in chapter 1 we have:

$$(\pi_1^* F_1, \pi_1^* F_2) = (F_1, F_2)$$

Using III.6 and III.7 it's clear that if (*) holds for S, then (*) holds for S_N. It remains to show that: if (*) holds for S_N, then it holds for $\hat{O}$. Assume that (*) holds for S_N. Let π_2 be the projection from $C \otimes S_N$ to $C \otimes \hat{O}$. The prime divisors C_p^i may split we lifted up to $C \otimes S_N$. Let

$$\pi_2^* C_p^i = \sum_j n^{ij} C^{ij},$$

where the C^{ij}'s are prime divisors of $C \otimes S_N$, and distinct divisors appear once in the summation. Since S_N is un- ramified over $\hat{O}$, we have $n^{ij} = 1$ for all i and j.

Let $\hat{K}$ be the fraction field of $\hat{O}$. Let D be a degree zero divisor of $C \otimes \hat{K}$, and let $\bar{D}$ be its closure in $C \otimes \hat{O}$.

Let L_N be the fraction field of S_N. Then $\pi_2^* \bar{D}$ will be the closure in $C \otimes S_N$, of some degree zero divisor of $C \otimes L_N$. Therefore, if (*) holds for S_N we have:

$$(m\pi_2^* \bar{D} + \sum d^{ij} C^{ij})^{\sigma}, E) = 0,$$

for all divisors E lying over $q_{N'}$ and for some integers m,$\{d^{ij}\}$. Now L_N is galois over $\hat{K}$ with galois group $\Gamma \simeq Z/NZ$, and Γ acts on $C \otimes S_N$. Furthermore the divisor $\pi_2^* \bar{D}$ is invariant under this action. Therefore, if $\sigma \in \Gamma$, we have:

$$(m\pi_2^* \bar{D} + (\sum d^{ij} C^{ij})^{\sigma}, E) = 0.$$

for all divisors E of $C \otimes S_N$ which lie over q_N. Sub tracting III.10 from III.9 we obtain:

$$(\sum d^{ij} C^{ij} - (\sum d^{ij} C^{ij})^{\sigma}, E) = 0,$$

for all E lying over q_N. This implies that the divisor $(\sum d^{ij} C^{ij}) - (\sum d^{ij} C^{ij})^{\sigma}$ is a multiple of the closed fiber of $C \otimes S_N$: If x is a uniformizing parameter of $S_{N'}$ than (x) is a principal divisor of $C \otimes S_N$ and:

$$(\sum d^{ij} C^{ij}) - (\sum d^{ij} C^{ij})^{\sigma} = n(\sigma)(x),$$

for some integer $n(\sigma)$. If σ' is another element of Γ and we let σ' act on III.12 we obtain

$$(\sum d^{ij} C^{ij})^{\sigma'} - (\sum d^{ij} C^{ij})^{\sigma+\sigma'} = n(\sigma)(x).$$

If we substitute σ' for a in III.12 and add the resulting equation to III.13 we obtain:

$$(\sum d^{ij} C^{ij}) = (\sum d^{ij} C^{ij})^{\sigma+\sigma'} = (n(\sigma) + n(\sigma'))(x).$$

We therefore have:

$$n(\sigma+\sigma') = n(\sigma) + n(\sigma').$$

Since Γ is a finite cyclic group, we have $n(\sigma) = 0$ for all $\sigma \in \Gamma$. Therefore

$$(\sum d^{ij} C^{ij}) = (\sum d^{ij} C^{ij})^{\sigma},$$

for all $\sigma \in \Gamma$. The action of Γ on $C \otimes S_N$ induces a transitive action on the set $\{C^{ij}\}_j$. We therefore have

$$d^{ij} = d^{ij'},$$

for all j and j'. Consequently the divisor $\sum d^{ij} C^{ij}$ is the pullback, via π_2, of some divisor $\sum d^i C_p^i$ of $C \otimes \hat{O}$:

$$\sum d^{ij} C^{ij} = \pi_2^* (\sum d^i C_p^i).$$

Let E be a divisor of $C \otimes \hat{O}$ which lies over p, we have:

$$(m\bar{D} + \sum d^i C_p^i, E) = (\pi_2^*(m\bar{D} + \sum d^i C_p^i), \pi_2^* E) = 0.$$

Therefore, if (*) holds for the ring R = S, the proof of proposition 1 will be complete.

In order to show that (*) holds for the ring S we will review Raynaud's work on Picard Functors.

Let q be the maximal ideal of S, L fraction field of S, and $k(q) = k(\bar{p})$ the residue field of S. We note that $C \otimes S$ is a regular curve over S. Let J_L be the Jacobian variety of $C \otimes L$. Let P_S be the Picard functor of $C \otimes S$, P_S is representable by a group scheme over.

Let $P_L = P_S \otimes L,$ the Picard scheme of $C \otimes L.$

Let e_L be the identity point of P_L and let ε_S be the scheme-theoretic closure of e_L in P_s. Let Q_s be the quotient of P_s by $\varepsilon_S.$ The group scheme Q_s is smooth, separated, and locally of finite type over S, see [Re]. Since $\varepsilon_S \otimes L = e_L,$ we have:

$$Q_S \otimes L = P_L,$$

in fact Q_s is the Neron model of P_L over S, see [Re]. Since J_L is a subscheme of P_L, it is also a subscheme of Q_L. Let Q'_s be the scheme-theoretic closure of J_L in Q_s. Then Q'_s is the Neron model of J_L over S,

see [Re].

Let Δ be the group of divisors of $C \otimes S$ which lie over q, and let Δ_0 be the subgroup of principal divisors. We have:

$$\varepsilon_S(s) \simeq \Delta/\Delta_0,$$

see [Re]. Consequently we have an exact sequence:

$$\Delta \to P_S(s) \to Q_S(s) \to 0.$$

Let $P'_S(s)$ be the inverse image, in $P_S(s)$, of $Q'_S(s)$ An invertible sheaf of $C \otimes S$ is in $P'_S(S)$ if the restriction of that sheaf to $C \otimes L$ has degree zero. We note that every element of P_s (S) can be represented by an invertible sheaf of $C \otimes S$ since C satisfies III.2, see [Re] .

We must define certain subschemes of P_s and Q_s : Let G be a commutative group scheme over S. Let G^o_q (resp. G^o_L) be the connected component which contains the identity of the group scheme $G \otimes k(q)$ (resp. $G \otimes L$) Let G^o be the subgroup scheme of G defined as follows:

Let $T \to G$ be a morphism over S, tnis morphism factors through G^o if the induced morphisms $T \otimes k(q) \to G \otimes k(q)$ and $T \otimes L \to G \otimes L$ factor through G^o_q (resp. G^o_L). Let G^τ be the subgroup scheme of G defined as follows: An s-morphism $f: T \to G$ factors through G^τ if some multiple of $f,\ f^n: T \to G,$ factors through G^o. In this manner we define group schemes $Q^o_S,\ Q^\tau_S,\ P^o_S,\ P^\tau_S.$ We collect some facts about these schemes: An invertible sheaf of $C \otimes S$ corresponds to an element of $P^o_S(S)$ if the degree of that sheaf, when restricted to each prime divisor of $C \otimes S$ which lies over q, has degree zero. Since $Q^\tau_S \otimes L \simeq Q'_S \otimes L$ and Q^τ_S is closed in Q_s (see [Re]) we have: $Q^\tau_S \simeq Q'_S.$ Finally Raynaud shows that the fact that III.2 holds for C implies that the map $P^o_S(S) \to Q^o_S(S)$ *is* injective, see [Re].

Let $C'_q, \ldots, C^r_q$ be the prime divisors of $C \otimes S$ which lie over q. We define a map $\alpha: \Delta \to Z^r$ as follows:

Let (,) be the Lichtenbaum-Shaferavitch pairing on $C \otimes S.$ Let E be an element of $\Delta,$ and let

$\alpha(E) = ((E, C^1_q), \ldots, (E, C^r_q)).$ Let n_i be the multiplicity of $C \otimes k(q)$ along C^i_q: if X_i is the generic point of C^i_q then

$$n_i = \text{length } (O_{X_i, C\otimes S}).$$

We define a map $\beta: Z^r \to Z$ by

$$\beta(a_1, \ldots, a_r) = \sum_i n_i a_i.$$

Since ((q),E) = 0 for all E in Δ we have $\beta \circ \alpha = 0.$ Let Δ^* be the kernel of $\beta.$ We have:

$$\Delta^{*}/\mathrm{Im}\ \alpha \simeq Q_S^{'}(S)/Q_S^{O}(S),$$

by [Re]. Recall that c -Try element of P_s (S) can be represented by a divisor E of $C \otimes S$. Define a map $\rho: P_S(S) \to Z^r$ by

$$\rho(E) = ((E,C_q^{'}),\ldots,(E,C_q^{r})).$$

Let E be a divisor of $C \otimes S$, and let E_L be the induced divisor of $C \otimes L$. Since the degree function $P_S(S) \to Z$ is locally constant we have:

$$\text{degree of } E_L = \sum_i n_i(E,C_q^{i}).$$

A divisor E of $C \otimes S$ will represent an element of $P_S^{'}(S)$ if the degree of E_L is zero. Consequently the map ρ above will restrict to a map

$$\rho: P_S^{'}(S) \to \Delta^{*}$$

By [Re], the map $\rho: P_S(S) \to Z^r$ is subjective, hence 11.29 is subjective. We summarize these results in the following diagram, here all rows and columns are exact and all squares are commutative:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & P_S^{O}(S) & \to & Q_S^{O}(S) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Delta & \to & P_S^{'}(S) & \to & Q_S^{'}(S) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\Delta & \to & \Delta^{*} & \to & Q_S^{'}(S)/Q_S^{O}(S) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Proposition 2: The condition (*) holds for the ring S.

Proof: Let D be a divisor of $C \otimes L,$ of degree zero, D represents an element [D] of J_L (L). Since $Q_S^{'}$ is the Neron model of J_L over S, [D] may be considered as an element of $Q_S^{'}(S)$. By the diagram above we may also consider [D] as an element of $P_S^{'}(S)/\Delta,$ i.e. [D] is just the calss of $\overline{D}$ in Pic $(C \otimes S)/\Delta$. Now $Q_S^{'}(S)/Q_S^{O}(S)$ is just a finite group, say with m elements. Therefore m[D], considered as an element of $Q_S^{'}(S)$ maps to the identity in $Q_S^{'}(S)/Q_S^{O}(S)$. Hence m[D] lies in the image of $Q_S^{O}(S)$ in $Q_S^{'}(S)$. Let γ be the element of $Q_S^{O}(S)$ which maps to m[D]. Hence $Q_S^{O}(S) \simeq P_S^{O}(S)$, we see that γ can be represented by a divisor E of $C \otimes S$, such that $(E,C_q^{i}) = 0$ for all i. Clearly E is linearly equivalent to $m\overline{D} + F$ for some F in $\Delta.$ In other words (*) holds for S.

Corollary: Let J be the Jacobian variety of $C \otimes K,$ and lry N be the Neron model of J over 0. Then the m in Proposition 1 is just the number of connected components of $N \otimes k(p)$.

Proof: We have $N \otimes S \simeq Q_S^{'}$ since Neron models are unique up to isomorphism. Furthermore the number of connected components of a smooth group scheme over k(p) is invariant under base extension by $\overline{k(p)}.$

REMARKS:

1. We note, for a given degree zero divisor D of $C \otimes K,$ the divisor $\sum d_p^i C_p^i$ in Proposition 1 is determined up to a multiple of the closed fiber of C which lies over p.

2. Let f be a nonzero rational function on C, let f_k be the restriction of f to $C \otimes K.$ Let $D = (f_K)$, a divisor of $C \otimes K,$ and let $\sum d_p^i C_p^i$ be as in Proposition 1.

Then we have:

$$\sum_i d_p^i C_p^i = m \sum_i \nu_p^i(f; C_p^i + n(p)$$

Where $\nu_p^i(f)$ is the order of f along C_p^i, (p) is the divisor corresponding to the entire fiber of C over p, and n is some integer. Let D and E be degree zero divisors of $C \otimes K$. Let $\overline{D}$ (resp. $\overline{E}$) be the closure of D (resp. E) in C.

Let p be a maximal ideal of 0, let $m_p, \{d_p^i\}$ correspond to $\overline{D}$ as in Proposition 1. Let

$$\langle D,E\rangle = [\overline{D},\overline{E}] + \sum_p \frac{1}{m_p}[\sum_i d_p^i C_p^i, \overline{E}],$$

the summation running over maximal ideals p of 0 such that $C \otimes k(p)$ has more than one component. Let Div° $C \otimes K$ be the group of degree zero divisors of $C \otimes K$.

Proposition 3: The pairing < , > is a bilinear, symmetric, real valued form on Div° $C \otimes K$, and its kernel contains the group of principal divisors of $C \otimes K$.

Proof: Let D' be another degree zero divisor of $C \otimes K$.

Let $\{e_p^i\}$ be a set of integers such that

$$[m_p\overline{D}' + \sum_i e_p^i C_p^i, F] = 0$$

for all divisors F lying over p. Then

$$[m_p(\overline{D}+\overline{D}') + \sum_i (d_p^i + d_p^i) C_p^i, F] = 0,$$

for all such F. Since [,] is bilinear, we conclude that < , > is linear in the first variable.

Let $\{f_p^i\}$ be the set of integers such that:

$$[m_p\overline{E} + \sum_i f_p^i C_p^i, F] = 0,$$

for all F lying over p. Then we have

$$[m\overline{D} + \sum d_p^i C_p^i, m\overline{E}]$$

$$= [m\overline{D} + \sum d_p^i C_p^i, m\overline{E} + f_p^i C_p^i]$$

$$= [m\overline{D}, m\overline{E} + \sum f_p^i C_p^i].$$

Symmetry for < , > and hence linearity in the second variable follow.

The last claim follows from the remarks following the proof of Proposition 2.

Let x be a closed point of C which lies over a maximal ideal p of 0. Let $C' \xrightarrow{\pi} C$ be the blowup of C at x. let < , > (resp. < , >') be the pairing on Div° $C \otimes K$ (resp. Div° $C' \otimes K$) defined by III.32. We are going to compare < , > and < , >'.

Let D and E be degree zero divisors of $C \otimes K$ Let D' (resp;. E') be the divisor of $C' \otimes K$ induced by D (resp. E). Let $\{C_p^i\}$ be the set of prime divisors of C which lie over p. Let F be exceptional divisor of C' over C. Let r (resp. r_i) be the multiplicity of x on ^{D} (resp. r_i) so

$$\pi^*\overline{D} = \overline{D'} + rF$$

$$\pi^* C_p^i = F_i + r_i F,$$

Where F_i is the strict transform of C_p^i. The set $\{F_i\} \cup \{F\}$ is the set of prime divisors of C' which lie over p. Let m_p, $\{d_p^i\}$ be a set of integers which satisfy:

$$[m_p\overline{D} + \sum_i d_p^i C_p^i, G] = 0,$$

for all divisors G of C which lie over p. Let

$$e = (\sum_i d_p^i r_i) + m_p r.$$

A simple calculation shows: Proposition 4:

$$[m_p \overline{D'} + \sum_i d_p^i F_i + eF, G] = 0,$$

for all divisors G of C* which lie over p. We have

$$\pi^*(m_p \overline{D} + \sum_i d_p^i C_p^i) = m_p \overline{D'} + \sum_i d_p^i F_i + eF.$$

Therefore, if we apply Proposition 9 of Chapter I, we obtain Proposition 5:

$$<D',E'>' = <D,E>.$$

Let C" be any regular curve over 0 such that $C'' \otimes K \simeq C \otimes K$. We may construct a pairing < , > on $C \otimes K$ by using Arakelov's intersection pairing on C". We may conclude that < , > agrees with < , > Indeed, by [Li] there exists a minimal model for $C \otimes K$ over 0. Both C and c" are obtained from this model from a finite sequence of blowups. By [Re], C satisfies III.2 if and only if the minimal satisfies III.2. By Proposition 5, the pairings < , > and < , > both agree with the pairing obtained from the minimal model.

Let K* be a finite extension of K and let 0' be the ring of integers of K'. Let C be as in the previous section. The scheme $C \otimes O'$ may not be regular: If C is not smooth over p, p maximal in 0, and p ramifies in o', then $C \otimes O'$ probably won't be regular at some points lying over p. However, by a finite sequence of blowups and normalizations we can desingularize $C \otimes O'$, obtaining a regular curve C* over 0', see [Aby] or [Lip]. Furthermore by [Re], C' will also satisfy III.2. Let $\pi: C' \to C$ be the composition of the desingularization $C' \to C \otimes O'$ and the projection $C \otimes O' \to C$. The map π is a finite morphism away from a finite set of closed points of C.

Let D and E be degree zero divisors of $C \otimes K$.

Let D' (resp. E') be the pullback of D (resp. E) to $C' \otimes K'$. Let < D, E > (resp.<D', E'>) be the pairing defined by III.32.

PROPOSITION 6:

$$[K':K]<D,E> = <D',E'>'.$$

Where [K' : K] is the degree of K' over K. The proof of this proposition will be given locally, one prime of 0 at a time.

Let p be a maximal ideal of 0 and let D_1 and D_2 be Weil divisors of C which meet properly. Let

$$[D_1,D_2]_p = \sum_x (D_1,D_2)_x \log |k(x)|,$$

where the summation runs over points x of C which are common to the supports of both D_1 and D_2' and which lie over p. Let D and E be two estranged degree zero divisors of $C \otimes K$. Let $\{C_p^i\}$ be the set of prime divisors of C which lie over p. Let $m_p, \{d_p^i\}$ be the set of integers such that

$$[m_p \overline{D} + \sum_i d_p^i C_p^i, F] = 0,$$

for all divisors F of C which lie over p. Let

$$<D,E>_p = [\overline{D},\overline{E}]_p + \frac{1}{m_p}[\sum_i d_p^i C_p^i, \overline{E}].$$

The pairing < , > is bilinear and symmetric whenever it is defined. Let D' (resp. E') be the pullback of D (resp. E) to $C' \otimes K'$. Then D' and E' are estranged degree zero divisors of $C' \otimes K'$. Let q be a maximal ideal of o', let <D', E'> be defined in the same fashion as III.43. Let

$$\chi_p(D,E) = [K':K]<D,E>_p - \sum_{q|p} <D',E'>_q,$$

which are common to the supports of both D_1 and D_2, and which lie over p.

Let D and E be two degree zero divisors of the summation running over maximal ideals q of O' which lie over p. If D is a principal divisor of $C \otimes K$, then an application of Proposition 9 of Chapter I shows that $\chi_p(D,E) = 0$. We may therefore define a pairing χ_p (,) on all of $C \otimes K$: If E and F are degree zero divisors of $C \otimes K$ we may find a divisor D of $C \otimes K$ which is linearly equivalent to F and estranged from E. Let

$$\chi_p(E,F) = \chi_p(E,D),$$

Where $\chi_p(E,D)$ is defined by III.44.

To prove Proposition 6 we will prove the Lemma. We have:

$$\chi_p(D,E) = 0,$$

for all degree zero divisors D and E of $C \otimes K$. Proof of lemma: Fix a divisor D for the rest of the discussion. We have

$$\chi_p(D,nE) = n\chi_p(D,E)$$

for all integers n, all degree zero divisors E. We will show that the left side of III.46 is bounded by a constant that depends only on D. This implies that

$$\chi_p(D,E) = 0.$$

We may assume that the support of $\overline{D}$ lies in the open subset U of C over which π is a finite morphism. Indeed C-U is a finite set of closed points, which lie in affine open subscheme of C since C is projective. By the Chinese Remainder theorem, there is a Weil divisor F of C which is linearly equivalent to $\overline{D}$ and whose support lies in U. Let G be the restriction of F to $C \otimes K$. Since the support of F lies in U, so does the support of $\overline{G}$. Since F is linearly equivalent to $\overline{D}$, G is linearly equivalent to D. Therefore

$$\chi_p(D,E) = \chi_p(G,E),$$

for all E.

Let F be an effective divisor of $C \otimes K$ which is which is estranged from D with deg $F \geq 2g + 1$. Then F is very ample. Let E be any decree zero divisor of $C \otimes K$ and let n be any positive integer. Then deg(F-nE) = deg $F \geq 2g + 1$, so F- nE is very ample. Therefore, there exists an effective divisor F_n of $C \otimes K$ which is estranged from D, such that nE is linearly equivalent to F-F_n. We therefore have:

$$\chi_p(D,nE) = \chi_p(D,F-F_n)$$

We claim that $\langle D, F-F_n\rangle_p$ differs from $[\overline{D},\overline{F}-\overline{F}_n]_p$ by a number whose absolute value is less than

$$\frac{\#\{C_p^i\}}{m_p}(2 \deg F)(\max_i |d_p^i|)(\log |k(p)|$$

Indeed, let x be a closed point of $C \otimes K$, $\overline{X}$ Its closure in C. Let n_p^i be the multiplicity of $C \otimes K$ (p) along C_p^i, see 11.24. A simple analysis shows that

$$[\sum_i n_p^i C_p^i, \overline{x}] \leq [K(x):K] \cdot \log |k(p)|$$

where k(x) is the residue field of x and [K(x):K] is the degree of K(x) over K. The claim made in the beginning of the paragraph follows.

An identical argument shows that $\sum_{q|p} \langle D', F'-F_n'\rangle_q'$ differs from $\sum_{q|p} [D', F'-F_n]_q$ by a number whose absolute value is bounded by a constant depending only on D.

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