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Grid Generation Methods and Techniques in Partial Differential Equation

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The simplest place to start an exposition of the basic philosophy behind the use of an adapted, irregular grid is in one dimension. The most widely used method is the equidistributed mesh. The principles of the method were later applied to generating efficient computational grids for the numerical solution of steady PDEs. For example White used a transformation to arc-length coordinates to generate equidistributing meshes for the numerical solution of two-point boundary value problems. In another approach where the one-dimensional mesh was iterated by trying to reduce the truncation error of the solution of the underlying PDE after each iteration. This is a convenient point at which to formally introduce and dene the equidistribution principle.

The main strategy behind the equidistribution idea is quite self-explanatory. The idea is to choose a mesh such that a measure of either the geometry of the represented function, or of the error of the numerical solution, is distributed equally between adjacent nodes. This measure is prescribed via a user-defined function known as the monitor, a positive-definite function of the solution *u* and/or its derivatives $u_x, u_{xx},$ of the form.

$$M = M(x, u, u_x, u_{xx}).$$

Later on in this section, we shall introduce various choices of monitor function and illustrate their effect on the resulting mesh. However, we begin by stating how this measure is distributed over the grid in a formal definition.

Given a mesh representing a physical space in one-dimension $x \in [a, b]$ with $N+1$ mesh points $x_i, i = 0, \ldots, N,$ such that $x_0 = a$ and $x_N = b,$ the equidistribution principle can be written

$$\int_{x_i}^{x_{i+1}} M dx = \frac{1}{N} \int_a^b M dx \quad i = 0, \ldots, N-1. \tag{2.2}$$

However, in most grid generation applications it is often more convenient to think of the equidistribution idea as one of a coordinate mapping from a computational, space to a physical one. The goal of the grid generation problem then becomes one of finding a suitable coordinate mapping or transform. This approach is common and forms the basis of most grid generation techniques. and, indeed, moving mesh methods. Concentrating still on one dimension, we define the computational space $\xi \in [0, 1]$, so that the mesh points in physical space are related to the (usually regularly spaced) grid points ξ_i in the computational domain. Written formally, x is then a mapping from ξ to x

$$x = x(\xi).$$

Within this framework the equidistribution idea is written as

$$\int_0^{x(\xi_i)} M d\tilde{x} = \xi_i \int_0^1 M d\tilde{x} \tag{2.3}$$

or

$$\int_{x(\xi_i)}^{x(\xi_{i+1})} M d\tilde{x} = \frac{1}{N} \int_0^1 M d\tilde{x}. \tag{2.4}$$

Differentiating (2.3) with respect to ξ once gives the equation differentiating yet again yields the equation.

$$\frac{\partial}{\partial \xi}\left(M\frac{\partial x}{\partial \xi}\right) = 0. \qquad (2.5)$$

Following this approach, the solution of (2.5) with Dirichlet boundary conditions

$$x(0) = a \qquad\qquad x(1) = b$$

produces an equidistributed grid for the given monitor function However, equation (2.5) is non-linear since M depends not only on x but also on the solution u. To overcome this, an iterative approach is suggested using the algorithm

$$(M(x^p)x_{\xi}^{p+1})_{\xi} = 0 \qquad\qquad (p = 0, 1, ...)$$

which may be discretised in a semi-implicit style as follows.

$$M(x^p_{i+\frac{1}{2}})(x^{p+1}_{i+1} - x^{p+1}_i) - M(x^p_{i-\frac{1}{2}})(x^{p+1}_i - x^{p+1}_{i-1}) = 0. \qquad (2.6)$$

The resulting tridiagonal system is easily solved using, for example, a Jacobi-iteration method.

When generating an equidistributed grid for good representation of a function or initial condition, the values of the monitor are known exactly and the iteration is usually quick and successful. However when using this type of iterative process for adapting a mesh to give a better numerical solution to an underlying differential equation, it is common to use an interleaving approach where the grid and solution are alternately updated, with the solution being interpolated between changing states of the mesh.

We now consider a few examples of possible monitor functions. The simplest such monitor, M =1 produces an uniform equi-spaced grid. This monitor has been used in a moving mesh method with a moving boundary, as it permits attractive theoretical properties of the solution within the mesh movement, (details of which shall be discussed later in Section (2.3)). Elsewhere, early work showed that minimising the error between a numerical approximation over a computational cell was equivalent to equidistributing the curvature monitor raised to a specific power, depending on which error norm was considered. However, the most common desired feature of using the technique of equidistribution is that the resulting grids have high mesh resolutions where solution gradients are steep and lower resolutions where the solution is less active. This in turn implies that the grid will then provide good approximations of derivative terms when using a suitable numerical scheme or solver. For this reason it is common for the monitor to involve derivative terms of the solution u. In this case, the simplest idea is to use the first derivative of u with respect to x, i.e.

$$M = |u_x|.$$

The effect of the gradient monitor on a monotonic function is that the solution values themselves become equi-spaced, since.

$$\int_{x_i} x_{i+1} M dx = u_{i+1} - u_i$$

see Figure 2.1 The most popular choice of monitor is the arc-length of the solution which has been used in many mesh generation and moving mesh methods. The arc-length monitor is written as

$$M = \sqrt{1 + u_x^2}. \qquad (2.8)$$

This monitor gives a smoother mesh overall than the gradient monitor especially when encountering large variations in u, as shown in Figure 2.1

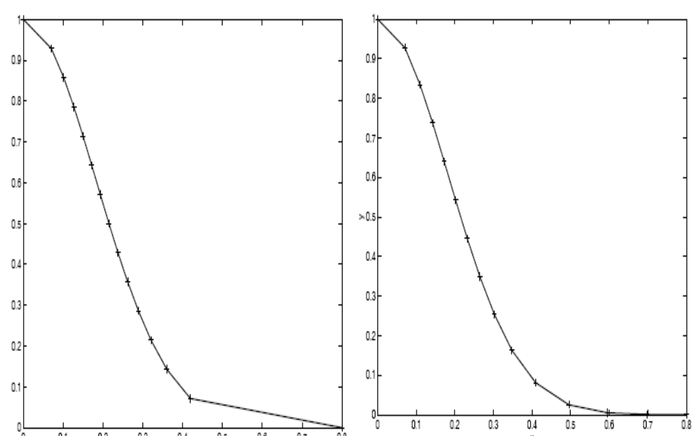

Figure 2.1 Examples of Grids using the gradient monitor 2.7 (left) and the arc- length monitor 2.8 (right).

In practice the derivative term in the arclength monitor is often scaled by some parameter α, for example

$$M = \sqrt{1 + \alpha u_x^2}. \qquad (2.9)$$

We shall comment more on the choice of monitor functions later on in Section 2.3.

Although equidistribution is the most common tool used when generating irregular computational meshes in one-dimension, the principle does not however extend strictly into twodimensions and an alternative is needed.

One of the earliest and most celebrated of such grid generation approaches in two dimensions is given in the appendix of Winslow's paper. The main body of which contains a method for the solution of a quasi-linear Poisson equation on a non-uniform triangular mesh, and the accompanying appendix outlines how to form such a mesh for regular domains. The ideas presented in this paper provide a basis for many of the higher-dimensional grid generation methods that followed. Once again, the approach is based on a mapping from a computational domain Ω_c to a physical domain Ω_p. The computational domain is represented as a regular equilateral triangular mesh composed of 2 sets of straight lines associated with the inverse mappings $\xi(x, y) \text{ and } \eta(x, y)$ which satisfy the Laplace equations

$$\nabla^2 \xi = 0, \qquad (2.10)$$

$$\nabla^2 \eta = 0. \qquad (2.11)$$

The solution to (2.10 and 2.11) results in intersecting equi-potentials, i.e. $\xi =$ constant and n = constant, with the mesh completed using the intersections of the resulting sets of lines. The required mesh is found by inverting the transforms and putting them in terms of $x(\xi, \eta) \text{ and } y(\xi, \eta)$ using the Jacobian $J = x_\eta y_\xi - x_\xi y_\eta$, so that (2.10 and 2.11) become

$$\alpha x_{\xi\xi} - 2\beta x_{\xi\eta} + \gamma x_{\eta\eta} = 0, \qquad (2.12)$$

$$\alpha y_{\xi\xi} - 2\beta y_{\xi\eta} + \gamma y_{\eta\eta} = 0 \qquad (2.13)$$

where

$$\begin{aligned} \alpha &= (x_\eta^2 + y_\eta^2), \\ \beta &= (x_\xi x_\eta + y_\xi y_\eta), \\ \gamma &= (x_\xi^2 + y_\xi^2). \end{aligned}$$

These equations can be discretised by the finite difference method outlined in the main body of the Winslow article and solved via an iterative successive over, relaxation algorithm. Due to the averaging property of the Laplace equation the constructed mesh is in some sense smooth and is also easily applicable to quadrilateral meshes. Notice that the method is in no-way linked to a function or numerical solution represented on the grid. The purpose of this early grid generation algorithm is to produce grids adapted to a particular domain, the shape of which is imposed via boundary conditions used in conjunction with (2.12 and 2.13) Winslow's method as outlined above was adopted by Thompson et al to generate meshes around multiple curvilinear bodies used in modeling flow over various shaped airfoils.

Brackbill and Saltzman took advantage of the idea and extended the method by allowing discretionary control of various mesh properties such as the smoothness and the orthogonality of the grid. Their paper highlights that solving the Laplace equations (2.10 and 2.11) is equivalent to minimizing the functional (2.14) below

which relates to the smoothness of the mesh, over the computational domain Ω_c.

$$I_s = \int_{\Omega_c} [(\nabla\xi)^2 + (\nabla\eta)^2] dV. \qquad (2.14)$$

Similarly, by solving the Euler equations associated with minimizing the functional related to the orthogonality of the mesh, (see (2.15)) an orthogonal grid is produced.

$$I_o = \int_{\Omega_c} (\nabla\xi . \nabla\eta)^2 dV. \qquad (2.15)$$

In practice, Brackbill and Saltzman suggested the use of linear combinations of such functionals, with the preferences of the user implemented through choices of coefficients. The overriding theme seems to be that as such properties can be measured they can also be controlled. In their paper the variational approach was used in conjunction with a numerical solution to a steady PDE and results show that, as the chosen functional is minimized, so too is the numerical error. Hence we see the development of the idea of a choice of functional in higher dimensions mirroring the effect of a monitor function in one-dimensional equidistribution.

The methodology of Winslow and Brackbill & Saltzman can be thought of as special cases of a more general framework outlined later by Huang & Russell Specifically presents the following functional (2.16) as a general form of a grid adaptation functional.

$$I[\xi,\eta] = \frac{1}{2}\int_{\Omega_p} [\nabla\xi^T G_1^{-1}\nabla\xi + \nabla\eta^T G_2^{-1}\nabla\eta]dxdy \qquad (2.16)$$

where G_1 and G_2 are given symmetric positive definite matrices, referred to as the monitor functions. The desired mesh transformation is derived from the solution to the associated Euler-Lagrange equations.

$$\nabla.(G_1^{-1}\nabla\xi) = 0, \qquad (2.17)$$

$$\nabla.(G_2^{-1}\nabla\eta) = 0. \qquad (2.18)$$

It is easy to see that by choosing $G_1 = G_2 = I$ this general methodology reduces to Winslow's original ideas Moreover Huang & Russell give forms of G_1 and G_2 which correspond to Brackbill's mesh generation method .

Equations (2.17) and (2.18) together with Dirichlet boundary conditions form a harmonic map from the physical to computational domains and the reliability of the method stems from the guaranteed existence and uniqueness of the transform, provided that the boundary of Ω_p is convex and that $G_1 = G_2$. Details can be found in Dvinsky Again, are given, involving distances from a given surface. As an illustrative example, below is the 'arc-length-like'

$$G_1 = G_2 = \frac{1}{\sqrt{1+\|\nabla u\|^2}}(I + \nabla u \nabla u^T). \qquad (2.19)$$

Further work by Cao et al proved by the use of Green's functions that the mesh can be aligned in certain directions and mesh concentrations can also be influenced in certain directions by controlling the eigenvectors and eigenvalues of the monitor matrices (specifically when $G_1 = G_2$). In particular, findings from this paper suggest that minimising the function I concentrates nodes in regions where the eigenvectors of G_1, λ_1 and λ_2 change significantly. This seems to have stemmed from earlier work by Brackbill and Knupp The latter followed his own earlier work, this time combining the Winslow functional and another functional giving a certain amount of directional control over the grid by attempting to align mesh lines with a prescribed vector field related to the approximate solution. Knupp also used the variational approach to grid generation, using weights from sets of vector fields, with the resulting meshes aligning themselves with the same vector fields in some least-squares sense, of course some prior knowledge of the appropriate vector fields being needed.

Another interesting example of the application of Winslow and other such methods, is outlined in Farmer for use in modeling geological features. Here grids are needed which honour 'control lines' representing features such as faults. These control lines are extended to the boundaries of the domain via interpolation, leaving the domain sectioned into several rectangular domains, which are then discretised using the outlined grid generation techniques.

This functional framework for finding the desired mesh transformation is a popular and convenient one, especially when used as the basis of a higher dimensional moving method, as we shall see later. For completeness, it is worth noticing that in one dimension, minimising the functional

$$I[\xi] = \frac{1}{2}\int_0^1 \frac{1}{M}\left(\frac{\partial\xi}{\partial x}\right)^2 dx$$

yields the equidistribution equation for a given monitor M. Since this general framework has been developed to work as part of high dimensional moving methods solution procedures for these methods incorporating the mesh movement process will be outlined later in Section 2.2

Alternative two-dimensional analogues of equidistribution for grid generation can be found in Baines and Huang & Sloan. In the former paper an equation to solve for the appropriate monitor function is given as a natural generalisation of equation (2.5)

$$\nabla_\zeta(M(n)\nabla_\zeta n) = 0$$

where n is a coordinate along the direction of ∇u and $\zeta = (\xi, \eta)$. This translates into a 'local equidistribution in the direction of ∇u'. Replacing n by x or y gives the equations below, which are of the familiar Euler-Lagrange form presented earlier.

$$\nabla_\zeta(M\nabla_\zeta x) = 0,$$

$$\nabla_\zeta(M\nabla_\zeta y) = 0.$$

These equations are again solved with an interleaving approach with Dirichlet conditions. The resulting grid is unable to equidistribute M precisely but clusters grid points in regions of high M as desired. Further, Baines shows that a least squares minimization of a residual of a vector field is equivalent to a least squares measure of equidistribution on triangular meshes, in some sense extending the work in one dimension by Carey & Dinh.

Elsewhere, the work of Huang and Sloan follows ideas set out by Dwyer and Catherall and a local equidistribution is obtained by imposing the strict one-dimensional form over two sets of coordinate lines.

It is worth taking time to grasp an understanding of these grid generation techniques as a precursor to studying moving-mesh methods. As we shall see in the following section, many moving grid algorithms are based upon an underlying principle for constructing meshes with effective grid resolutions.

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