Probability: Settings and Rationales Why Probability

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THE SUBJECT EVERYONE LOVES TO HATE

Concepts and intuitions from probability theory are ubiquitous in many aspects of our "non-mathematical" lives. When we decide whether to pay more for a car that is reported to be "more reliable"; when we choose to safeguard our children by living in a "low-crime" neighborhood; when we change our diet and exercise habits to lower the risk of heart disease; when we accept or refuse a job with other options pending; all the above actions involve implicit probability judgments.

Arguably, no other area of mathematics can be said to apply so widely, to be so potentially useful in making decisions, both intimate and professional, and affords such opportunities to make sense of phenomena, both natural and social, and empower us through the knowledge we gain. Yet, despite this potential for empowerment, people's 90 attitude towards probability can be summed up in the well-worn adage: “There are three kinds of lies”, said Disraeli, “lies, damn lies, and statistics.”

Probability courses are anathema to most graduate students. Students in the social sciences are required to take a statistics course, and frequently report being able to work the textbook problems but having no idea “what they mean,” or how to apply them to novel situations (see, e.g., Phillips, 1988). They also say that they can make many different arguments to solve a problem and they all can sound plausible, but they give different answers -- how to choose among the plausible alternatives?

UNRESOLVED PHILOSOPHICAL UNDERPINNINGS

A second source of reasons to look more carefully at probability comes out of the philosophy of science. As we shall see below, the meanings of the basic notions of probability theory, the ideas of "randomness", "distribution", and "probability" itself are still quite controversial. We are living in a time when these meanings are not yet fixed but are being negotiated by mathematicians and philosophers. Due to the lack of cultural construction of basic probabilistic concepts, we are, as probabilistic learners, in a similar position to the pre-conservational child. As teachers of children, having the experience of conflicting probabilistic arguments that we cannot resolve, serves as an empathy pump, reminding us of how difficult the acquisition of new mathematical ideas can be, of the slow pace of conceptual change, and that a post-conservational perspective is inadequate for helping learners move beyond their pre-conservational conceptions.

EPISTEMOLOGY OF PROBABILITY

Many historians of science would say that the real period of "probabilistic revolution" occurred in the 1830's. (see, e.g., Hacking, 1990; Cohen, 1990) 91 Among the dominant interpretations of the meaning of probability we can identify four schools:

Propensitists: believe that probabilities are essential properties of objects. Just as a coin has mass, volume, and color, so it also has another property which is a propensity when thrown to land on heads 50% of the time and tails 50% of the time. Discovering these properties can be done in many different ways but is analogous to discovering other objective properties.

Frequentists: probabilities are limiting ratios of frequencies. When tossing a coin many times, we record the ratio of numbers of heads to number of tosses. As the number of tosses increases without bound, this ratio approaches the probability of throwing a head.

Subjectivists: probabilities are degrees of belief. When tossing a coin, the probability of its coming up heads is relative to the beliefs of the coin tosser. We can measure subjective degrees of belief by giving people bets at various odds and seeing which they judge to be fair bets. So, in assessing the degree of their belief in the proposition "the next toss of the coin will be heads", we can offer them the choice of taking $5 and "running" or taking $10 if the next toss is heads and zero if it's tails. If they judge this choice to be a toss-up (pun intended) then we say their degree of belief in the proposition is 1/2. If they'd rather have the $5, we say their degree of belief is less than 1/2.

credibilists or Bayesians: this view identifies probability with rational degree of belief given the evidence. Accordingly, probabilities are not just subjective degrees of belief, nor are they objective properties of objects, rather they are those degrees of belief which a rational person would hold given the evidence she has available. Since two people might have different evidence, they might have different probabilities for the same 92event. In judging the probability of a coin toss, one person seeing a streak of heads may 45 judge that the probability of the next toss being heads is still 1/2, while another may judge that the probability of heads is greater than 1/2. They could both have rational degrees of belief if one has evidence of the common circulation of biased coins and the other does not.

This controversy may seem rather theoretical, but consider what these differences mean for how we understand and apply probability to our daily lives.

If we are naive frequentists, then, for most situations in our life, probability is irrelevant. After all, probabilities are limiting ratios of frequencies. But, for an event to have a limiting frequency ratio, it must be repeatable. Usually, when we act in the world, we consider ourselves to be in a unique situation, one unlikely to exactly repeat itself.

Under this view, probability is irrelevant to most life situations. Only in a few narrow contexts, such as gambling, can the ideas of probability apply to our life decisions. Thus, the naive frequentist, to secure the bedrock of well defined probability and normative rules dictating “correct action”, sacrifices the connection of probability to most life contexts. Of course, a more sophisticated frequentist might argue that we can still apply probability to a unique situation if we can find the right level of description for that situation. If we find a description that identifies our unique event with other events and therefore forms a class of events, then we can treat the unique event as a repeatable event.

In this view, a probability is not a property of an object, but rather a property of our knowledge of the object.

Actually, the bedrock is not quite so solid. Even in "obviously" repeatable situations such as bets at roulette, the frequentist still must specify a criterion for identifying two situations. In order for your roulette bet to be repeatable, subsequent bets must be considered "the same". But in the next bet, someone new is standing next to you, the air movement is slightly different, ... The frequentist is forced to say that these differences don't make a difference and therefore the two bets are equivalent, but determining which variables can be relevant to considering two situations the same or different is a serious problem for the frequentist. The philosopher Nelson Goodman (1983) has shown that one cannot even be sure that a single predicate has the same value in two different situations. As we saw in Chapter IV, deciding when two situations or objects are the same is an act of construction by the individual involved, and different individuals will construct different "sameness" maps.

In contrast, the subjectivist allows all events to have probabilities. Each individual may assign any probability to any event. The price of the richness of application however, is the lack of guidance. Each of us may have a completely different probability for an event to occur. Which is right? How can we evaluate our probability, our beliefs?

At first glance, the Bayesians seem to have the best of both worlds above. On the one hand, they allow all events to have probabilities, and thus probability is relevant. On the other hand, the Bayesian procedure for updating our probabilities gives us rational guidance. Outlandish probabilities will not stand the test of the evidence. But there are problems with the Bayesian view as well. In order to "ground" the Bayesian update formula, each event must be assigned an initial or "a priori" probability. How is this probability to be chosen? One answer is known as the "principle of insufficient reason".

It says, that if you don't know how to assign probabilities, if you are "ignorant", then you should assign equal probabilities to all possibilities. But this answer is fraught with paradox. Consider the following example: You want to know if there is life in orbit around a star, say Mithrandir, in a faraway galaxy. Since you know nothing about Mithrandir, you decide to assign equal probability to the belief that there is life on Mithrandir as to its contrary. But now consider the question: are there planets surrounding Mithrandir? If, to this, we admit there are three possibilities:

P1: There is life orbiting Mithrandir.

P2: There are planets but no life.

P3: Mithrandir has neither planets nor life.

Then we are completely ignorant of which possibility obtains. By the principle of insufficient reason, we should then say that the probability of each event is equal to 1/3.

Therefore, the combination of P2 and P3 will now have probability 2/3 . But the combination of P2 and P3 correspond to the assertion that there is no life orbiting Mithrandir which, by the principle of insufficient reason, was assigned probability 1/2 before. In this manner we can create completely contradictory probability assignments and it thus appears that there is no rational way to choose our prior probabilities.

Bayesians have found various routes around this difficulty but none have been universally adopted. Furthermore, in order for Bayesians to be able to use their formula for updating probabilities, they must, like the frequentists, be able to determine when two events are to "count" as the same event. This too is fraught with difficulties.

I maintain that the confusion commonly experienced by first-year probability students is not unrelated to the confusion and controversy surrounding these core notions of probability. The difficulties Tversky and Kahneman report are not merely computational; they reflect a deep epistemological confusion as to what the quantities to be calculated are? If the latter conjecture is well-founded, then building a connected mathematics learning environment for probability might go a long way toward grounding students in the subject matter.

TVERSKY & KAHNEMAN

Yet a third source of motivation for my focus on probability is the research uncovered by Tversky and Kahneman (1982) on the persistent errors that people make when making judgments under uncertainty. Some of their findings have been outlined earlier in this thesis. Now, we will look at their results in greater detail.

The psychologists Tversky and Kahneman have spawned a vast literature which documents people's systematic biases in reasoning when reasoning about likelihoods.

Shafer (1976) finds an interesting way around this dilemma which results in a very different notion of probability.

I have recently come across fascinating research by Cliff Konold (e.g. Konold, 1989; 1991) which begins to suggest an affirmative answer to this question. In asking people questions such as "The weatherman predicts there's a 70% chance of rain tomorrow and it doesn't rain, what can you say about his prediction?", many subjects reported that the weatherman was wrong. According to Konold this shows they were understanding probability as a way of predicting the next single outcome. In doing that, they anchored all probabilities to three basic quantities: 0 = no possibility, 1 = certain, and 1/2 = don't know. In the weatherman case, 70% was interpreted as anchored to 1.

CONSIDER THE FOLLOWING EXAMPLE

Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.

Please rank the following statements by likelihood:

1) Linda is a bank teller.

2) Linda is active in the feminist movement.

3) Linda is a bank teller and is active in the feminist movement.

Tversky & Kahneman analyzed people's responses to questions of this type. A persistent “misconception” that respondents exhibited was to rank Linda's being a bank teller and a feminist as more likely than just being a banker. This ranking is in violation of the rules of logic which require that any statement is more likely than its conjunction with another statement. Yet, even when the respondents had sophisticated knowledge in logic and probability they were somewhat seduced by the incorrect answer - seeing it as intuitively right and trying to find some way to justify it.

Tversky & Kahneman explained this misconception as stemming from people's use of a "representativeness" heuristic in making likelihood judgments. By a "representativeness heuristic" they mean that people look for an ideal type that represents their answer and then judge probability by closeness to this type. In the example of Linda, the text leads us to represent Linda as a someone who is likely to be a feminist and unlikely to be a banker. (So, when we judge the likelihood of the three statements, we see Some researchers have compared this phenomenon to visual illusions such as the face/vase illusion.

They conceptualize Tversky & Kahnemann kinds of examples as "cognitive illusions". that it is unlikely that she is a banker, likely that she is a feminist, and yes perhaps she could be a banker because maybe it was the only job she could get, if she's still true to the feminist type.)

Tversky and Kahneman have collected many examples of this type. In one experiment, people are asked whether there are more words in the English language that begin with “r” or that have “r” as their third letter. Most people say there are more words that begin with “r” whereas in fact there are many more of the latter kind. In this case, Tversky and Kahneman argue that the error is attributable to the heuristic of "availability". People can much more easily "retrieve" or recall words that begin with “r” than words with “r” in the third position. Since the words beginning with “r” are more available to them, they judge them more likely. Another way to describe the availability heuristic is to think of people who are recalling words with “r” in it as conducting sampling experiments. They sample a few words with “r” in them and then compute the relative frequency of “r” in the first or third position. Under this interpretation, the error that people make is in wrongly attributing randomness to their procedures for generating samples of words with “r” in them.

Yet a third example of the errors Tversky & Kahneman report:

A group of “subjects” was given the following story:

A panel of psychologist have interviewed and administered personality tests to a group of 30 engineers and 70 lawyers, all successful in their respective fields. On the basis of this information, thumbnail descriptions of the 30 engineers and 70 lawyers have been written. You will find on the form five descriptions, chosen at random from the 100 available descriptions. For each description, please indicate your probability that each person is an engineer, on a scale from 0 to 100.

Subjects in another large group were given the same exact story except that there were 30 lawyers and 70 engineers. Both groups were then presented with 5 descriptions. For example:

Jack is a 45 year old man. He is married and has 4 children. He is generally conservative, careful, and ambitious. He shows no interest in political and social issues, and spends much of his free time on his many hobbies, which include home carpentry, sailing and mathematical puzzles.

Subjects in both groups judged Jack to be much more likely to be an engineer.

The data indicate that the prior probability or "base rate" of lawyers or engineers did not make an appreciable difference. But when the same subjects were asked to make the same judgments in the absence of a personality description, they did use the base rates.

Tversky & Kahneman concluded that in the presence of specific descriptions, prior probabilities are ignored.

Because these systematic errors are repeatable and don't seem to go away even when people have had significant training in probability, there is a widespread belief that humans are incapable of thinking intuitively about probability. As many tell the story, the human brain evolved at a time when probabilistically accurate judgments were not required and, consequently, resorted to heuristic shortcuts that were not so taxing on mental resources. As a result, we have been "hard-wired" not be able to think about probability and must circumvent our natural thinking processes in order to overcome this liability.

Tversky & Kahneman speculate as to the origin of the systematic biases they uncovered in people's assessment of likelihoods. They theorize that people's mental resources are too limited to be able to generate probabilistically accurate judgments.

People are forced to fall back on computationally simpler heuristics such as the representativeness heuristic they describe.

This view has become very influential and has spawned a large literature.

Interpreters of Tversky and Kahneman seem to come in two varieties: those who make the strong claim that our brains are simply not wired for doing probability, that evolution did not spend our mental resources so profligately, and those who simply hold the weaker claim that as far as probability is concerned, our intuitions are suspect. However, the effect of these claims on probability education has been the same -- a reliance on formal methods and a distrust of intuitions.

Recently, I took an introductory graduate course in probability and statistics.

When we came to the section on inverse probabilities, the professor wrote down Bayes theorem for calculating inverse probabilities and then baldly announced:

"Don't even try to do inverse probabilities in your head. Always use Bayes formula. As Tversky and Kahneman have shown, it is impossible for humans to get an intuitive feel for inverse probabilities".

Given the prevalence of this formalist view, it would be of great interest if it could be shown that these systematic probabilistic biases could be transformed into good probabilistic intuitions by a suitable learning environment.

And after all, we do have some reasons to doubt that the lack of robust intuitions about the meanings and applications of probabilistic concepts is due to some inherent deficiency in the "wiring" of the human brain. It may instead stem from a lack of concrete experiences from which these intuitions can develop. Rarely, in our everyday lives, do we have direct and controlled access to large numbers of experimental trials, measurements of large populations, or repeated assessments of likelihood with feedback. We do regularly assess the probability of specific events occurring. However, when the event either occurs or not, we don't know how to feed this result back into our original assessment. Suppose we assess the probability of some event occurring as say 30%, and the event occurs, we have not gotten much information about the adequacy of our original judgment. Only by repeated trials can we get the feedback we need to evaluate our judgments.

It is a plausible conjecture that the computer (with its large computational resources, capacity to repeat and vary large numbers of trials, ability to show the results of these trials in compressed time and often in visual form) may be an important aid in construction of a learning environment which gives learners the kinds of concrete experiences they need to build solid probabilistic intuitions.

In the following chapter, I will argue that, even though many of the empirical results of Tversky & Kahneman do hold for many people today, concluding from this fact that people are innately unable to think about probability is unwarranted. This argument will rest on two claims, one theoretical and one empirical. The theoretical claim which we discussed in Chapter IV is that people's mathematical intuitions are constructed, not innately given. Both the lack of good learning environments for probability and the cultural and epistemological confusion surrounding probability do not support the construction of good probabilistic intuitions. Personal and cultural development can lead to more sophisticated probabilistic intuitions and greater mathematical understanding. In the interviews presented in the next chapter, we see learners beginning this concretizing process and starting down the road toward development of strong, reliable probabilistic intuitions.

EMERGENT PHENOMENA

The fourth source directing the inquiry into probability comes from the new fields of systems theory, emergent dynamics, and artificial life. There has been a rash of publications over the past few years about the difficulties people (and scientists) have with understanding emergent phenomena - phenomena that are the result of the interaction of numerous distributed but locally interacting agents. Mitchel Resnick (1991) has written eloquently about these difficulties and postulated the existence of a "centralized mindset" - a globalized tendency to think in terms of top-down, centrally organized, single agent control structures. To help people go beyond this mindset, he designed the language *Logo, which allows the programmer to control thousands of graphic turtles and thousands of "patches" (i.e., small pieces of the environment on which the turtles move. The patches alone can be thought of as cells in a cellular automaton).

*Logo provides primitives for basic turtle and patch calculations as well as communication among turtles and patches and interactions between them.

One of the difficulties encountered when trying to understand emergent phenomena is that though the resultant pattern is often stably determined, the sequence of steps by which it is reached is not at all deterministic. In one of the examples from Resnick's (1992) doctoral dissertation, a *Logo program simulates the interactions of termites and wood chips. Before the program is run, the patches are seeded randomly with wood chips and termites are scattered randomly on the screen. After running the program, you see the termites picking up chips and putting down chips and after a few minutes piles of chips begin to take clear shape on the screen. How does the program work? If one is operating from a centralized mindset, one might explain the behavior in terms of a planner termite who organizes all the termites into teams and tells them each what to do to achieve this effect. But in fact the program works by two simple rules: at each time step, move randomly. If you come to a wood-chip and are carrying one, then I have added primitives to *Logo to facilitate working with probability and statistics. Drop it and turn around, if you come to a wood chip and aren't carrying one, then pick one up. At first glance this procedure doesn't seem to be right. You might object: "but the termites will start to make piles and then destroy them - this is no way to build up a set of piles."

A key insight into seeing why this works is to note that the number of piles can never increase. Since termites are always putting down chips on top of already made piles, they can never start a new pile. Since in the long run, the number of piles will decrease, the average pile size must increase and eventually a few large piles appear on the screen.

This example shows some of the characteristic features of emergent phenomena.

The overall eventual pattern of the piles can be said to be determined, but on each run of the program, the path that the termites take and the details of the pile formation are quite different.

Emergent phenomena are essentially probabilistic and statistical. Some interesting questions to ask are: Are the difficulties in understanding emergent phenomena due to their probabilistic character? Is there such a thing as a deterministic mindset by analogy with a centralized mindset? Is the change that transpires when we say someone understands probability a matter of incremental knowledge -- a mere mastering of subject matter in a new mathematical area? Or is the change more fundamental --

a global change in the entire way of looking at the world? And if there is such a thing as a probabilistic mindset, does getting there require a quantitative understanding of probability or are qualitative arguments akin to the ones we gave in the termite example sufficient, or even preferable? The beginnings of answers to these questions will emerge from the interviews discussed in the next chapter.

If the program is run long enough, the number of piles should reduce to 1. It is an interesting piece of mathematics to try to calculate how many iterations this should take. Indeed, one lesson we take from Minsky and Papert's (1969) "Perceptrons", is that, while it is highly desirable to have a qualitative understanding of algorithms like "termites", it is also important to understand their complexity.

PROBABILITY SETTING

The Probability research has been conducted in a variety of research settings. Seventeen in-depth interviews (typically lasting at least two hours each, and some lasting as long as eighteen hours face to face!) were conducted. The interviewees consisted of seven women and ten men, one high school student, four undergraduates, five graduate students, and seven post-graduates. The interview format was designed to elicit discussion on four main topics:

1) How to make sense of basic probabilistic notions such as randomness, distribution,

probability?

2) How to go about solving particular probability problems? The problems were chosen by me for their potential for raising fundamental issues and likelihood to link to new and related problems. Some were chosen due to their counter-intuitive and paradoxical nature.

3) How to interpret statistics from newspapers, articles or books. "How would you design a study to collect this statistic?" and

4) What is your attitude towards thinking probabilistically?

Interviews were not time limited: they were allowed to continue until they reached a natural conclusion. Protocols for the interview were flexible: a total of 13 separate topics had to be covered before the interview was done, but no rigid order was imposed by the interviewer. A goal of the interview design was to be broad, to get at the relationship of the interviewee to Probability in a variety of contexts and to explore psychological and social dimensions of this relationship as well as the mathematical dimensions. The interview was experienced by most interviewees as a conversation. Most often, the conversation grew out of the responses of the interviewee to the beginning question:

"What does the word probable mean in ordinary discourse, and is this meaning related to the mathematical disciplines of Probability & Statistics?" Some interviewees started to give their personal accounts of experiences with courses in Probability before the initial question was asked. This was then followed up and became the germ from which the interview sprouted. Along the way the interviewer introduced a variety of prepared questions at points where they seemed natural. Among the topics explored were the relationship of the interviewee to uncertainty (what his/her general tolerance is for uncertainty in decision making), whether the interviewee participated in gambling activities, as well as specific Probability problems such as the "Monty Hall" problem recently publicized in the NY Times. The interviewer also presented statistics from newspaper articles (e.g., the divorce rate in 1987 is 50%, the reliability of a certain form of birth control is 95%), and asked the interviewee to interpret the meaning of the statistic, how it might have been collected, what kind of study would he/she design in order to obtain this statistic.

Great effort was made by the interviewer to combat the inhibitions of the interviewees to talk about their partial and imperfect understandings. The interviewer explained his belief that all mathematical understanding is partial and a tangle of messy concepts. He also modeled the pushing through inhibition by talking about his own mathematical development and the many confusions and subsequent connections made on the road to mathematical understanding.

After the interview topics were covered, all interviewees expressed a desire to talk more about the problems and to "find out what the right answers were." At this point, the interviewer discussed his understanding of the solutions, and in cases where discourse did not settle the matter, experiments were designed by both parties and conducted by the interviewee in order to deepen the understanding of the problem.

In a few cases, the interviews could not be conducted (or completed) face to face either because the interviewee was no longer in the area or because the amount of time needed was more than could be arranged. Some of these interviewees elected to continue the interviews over electronic mail. In particular interviews on the "envelope paradox" were often conducted in this way. In email interviews, I sent each interviewee an initial message describing what I hoped for in our email dialogue. Most interviewees appeared to have no problem adhering to my guidelines and the email medium proved to allow a rich set of exchanges.

In addition to the interviews, some students worked with me in computer learning environments. Five students elected to develop *Logo programs designed to explore some area of probability that they wanted to understand better. In order to facilitate its use as an environment of exploring probability, I made some modifications to *Logo and added some primitives especially useful for probabilistic explorations. Among the programs developed are a microworld for exploring distributions of gas particles in a closed container, an environment for exploring the meaning of bell-shaped and binomial distributions, a test environment for "Monty Hall" type problems, and various projects modeling newspaper statistics, and creating physics and ecology simulations. The nature of these projects, what was learned through both their design and their use will be explored in detail.

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