The Deficient Discrete Quartic Spline over Uniform Mesh

Y. P. Dubey[1] Anil Shukla[2]

[1]Department of Mathematics, L.N.C.T. Jabalpur 482003

[2]Department of Mathematics, Gyan Ganga College of Technogy, Jabalpur

Abstract - In the present paper, we have studied the existence, uniqueness and convergence properties of discrete quartic spline interpolation over uniform mesh, which match the given functional values at mesh points, mid points and second derivative at boundary points.

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1. INTRODUCTION

Discrete splines have been introduced by Mangsarian and Schumaker [7] in connection with certain studied of minimization problem involving differences. Discrete cubic splines which interpolate given functional values at one points lying in each mesh interval of a uniform mesh have been studied in [2]. The case of these points coincide with the mesh points of a non uniform mesh was studied earlier by Lyche [5], [6]. To compute non-linear splines interactively Malecolm [3] used discrete splines. Mangasarian and Schumaker [8] used discrete splines for best summation formula. For some different constructive aspects of discrete splines, we refer to Schumaker [10], Astor and Duris [1], Jia [4] and Rana and Dubey [9]. In this paper we have obtained existence, uniqueness and convergence properties of dificient quartic spline interpolation over uniform mesh which matches the given functional values at mesh points and mid points with boundary condition of second difference.

Let us consider a mesh P on [a, b] which is defined by

$$P:0=x_0<x_1<......<x_n=b$$

For i=1,2,...n. P_i shall denote the length of the mesh interval $[x_{i-1},x_i]$, P is said to be a uniform mesh if P_i is constant for all i. Throughout, h will represent a given positive real number. Consider a real function *s(x, h)* defined over [0, 1] which is such that its restriction s_i on $[x_{i-1},x_i]$ is polynomial of degree 4 or less i=1, 2.....n. Then s(x, h) defines a deficient discrete quartic splines with deficiency 1 if

$$D_n^{\{j\}}S_i(x_i-h)=D_n^{\{j\}}S_{i+1}(x_i,h)$$
$$j=0,1,2 \tag{1.1}$$

Where the difference operator D_n^j for a function *f* is defined by

$$D_n^{\{0\}}f(x)=f(x),D_n^{\{1\}}f(x)=\frac{f(x+h)-f(x-h)}{2h}$$

$$D_n^{\{2\}}f(x)=\frac{\left(f(x+h)-2f(x)+f(x-h)\right)}{h^2}$$

and $=D_n^{(m+n)}f=D_n^{\{m\}}D_n^{\{n\}}f(x),\ m,n\geq 0$

The class of all deficient discrete quartic splines with deficiency 1 satisfying the boundary condition.

$$D_n^{\{2\}}f(x_0,h)=D_n^{\{2\}}f(x_0,h)$$

$$D_n^{\{2\}} f(x_n,h) = D_n^{\{2\}} f(x_n,h) \quad (1.2)$$

is denoted by R (4, 1, P, h).

Now writing $2\alpha_i = x_i + x_{i+1}$, we introduced the following interpolating condition for given function f.

$$s(x_i,h) = f(x_i,h) \qquad 1 = 0, 1,.....n \quad (1.3)$$

$$s(\alpha_i,h) = f(\alpha_i,h) \qquad i = 1,2,.....n$$

and pose the following.

PROBLEM : Given h > 0, for what restriction on P does there exist a unique s(x, h) ∈ R (4, 1, P, h) which satisfies the condition (1.2) and (1.3)?

2. EXISTENCE AND UNIQUENESS :

Let P(z) be a discrete quartic spline Polynomial on [0, 1], then we can show that

$$E(z) = E(0)R_1(z) + E(1)R_2(z) + E\left(\frac{1}{2}\right)R_3(z)$$

$$+ D_n^{\{2\}} E(0) R_4(z) + D_n^2 E(1) R_5(z) \quad (2.1)$$

Where

$$R_1(z) = \frac{1}{6A}\left[6A - (72h^2+78)z + 48h^2z^2 + 96z^3 - 48z^4\right]$$

$$R_2(z) = \frac{1}{6A}\left[(-24h^2-18)z + 48h^2z^2 + 96z^3 - 48z^4\right]$$

$$R_3(z) = \frac{1}{6A}\left[96(h^2+1)z - 96h^2z^2 - 192z^3 + 96z^4\right]$$

$$R_4(z) = \frac{1}{6A}\left[-(2h^2+4)z + 3(5+2h^2)z^2 - (17+4h^2)z^3 + 6z^4\right]$$

$$R_5(z) = \frac{1}{6A}\left[(2h^2+1)z - 6h^2z^2 + (4h^2-7)z^3 + 6z^4\right]$$

Where $A = \left[\frac{1}{4h^2+5}\right]$

Now we are set to answer problem A in the following.

Theorem 2.1. For h > 0, there exist a unique deficient discrete quartic spline s(x, h) ∈ R (4,1,P,h) which satisfies conditions (1.2) and (1.3).

Proof the Theorem 2.1 : Denoting $(x - x_i)$ by t, 0≤ t≤ 1. We can write (2.1) in the form of restriction $s_i(x,h)$ of the quartic spline $s(x,h)$ on $[x_i, x_{i+1}]$ as follows :-

$$s_i(x_i,h) = f(x_i)R_1(z) + f(x_{i+1})R_2(z)$$

$$+ f(\alpha_i)R_3(z) + P^2 D_n^{\{2\}} s(x_i,h) R_4(z)$$

$$+ P^2 R_5(z) D_n^{\{2\}} s(x_{i+1},h)$$

where

$$R_1(z) = \frac{1}{6AP^4}[6AP^4 - (72h^2+78)(x-x_i)P^3 + 48h^2(x-x_i)^2P^2$$

$$+ 96(x-x_i)P - 48(x-x_i)^4]$$

$$R_2(z) = \frac{1}{6AP^4}[(-24h^2-18)(x-x_i)P^3$$

$$+ 48h^2(x-x_i)^2P^2 + 96(x-x_i)^3P - 48(x-x\ )^4]$$

$$R_3(z) = \frac{1}{6AP^4}[96(h^2+1)(x-x_i)P^3 - 96h^2(x-x_i)^2P^2$$

$$-192(x-x_i)^3P + 96(x-x_i)^4]$$

$$R_4(z)=\frac{1}{6AP^4}[-12h^2+4)(x-x_i)P^3-(5+2h^2)$$

$$(x-x_i)^2P^2-(17+4h^2)(x-x_i)^3P\ +6(x-x_i)^4]$$

$$R_5(z)=\frac{1}{6AP^4}[(2h^2+1)(x-x_i)P^3-6h^2(x-x_i)^2P^2$$

$$+(4h^2-7)(x-x_i)^3P\ +6(x-x_i)^4]$$

Clearly $s_1(x,h)$ is a quartic on $[x_i, x_{i+1}$ for i=0,1,...n-1 and satisfies (1.2) and (1.3). Now applying continuity of first difference of $s_i(x,h)$ at x_i, given by (1,1), we get the following system of equation -

$$\left[(2h^2+1)P^2+(4h^2-7)h^2\right](D_h^{\{2\}}s_i(x_{i-1},h)$$

$$D_n^{\{2\}}s_i(x_i,h)2\left[(2h^2+4)P^2+(17+4h^2)h^2\right]$$

$$+D_n^{\{2\}}S(_{i+1},h)\left[(2h^2+1)P^2+(4h^2-7)h^2\right]$$

$$=F_i \qquad \text{i = 1,2,...n} \qquad \text{Say}$$

Where

$$F_i=\frac{1}{P^2}[\{(24h^2+18)P^2-96h^2\}(f(x_{i-1})+f(x_{i+1})$$

$$+2\{(72h^2-78)P^2-96h^2)f(x_i)+\{-96(h^2+1)+192h^2\}(f(\alpha_i)+f(\alpha_{i-1}))]$$

Write $D_n^{\{2\}}(x_i,h)=m_i(h)=m_i$ for all i.

Say

We can easily see that excess of the absolute value of the coefficient of m_i over the sum of the absolute value of the coefficient of m_{i-1} and m_{i+1} in (2.3) under the condition of theorem 2.1 is given by

$$d_i(h)=6(P^2+8h^2)$$

which is clearly positive.

Therefore the coefficient matrix of the system of equation (2.3) is diagonally dominant and hence invertible. Thus the system of equation has unique solution. This complete the proof of theorem 2.1.

3. ERROR BOUND

For a given h > 0, we introduce the set

$$R_n=\{jh : j\, is\, an\, \text{int}\, eger\}$$

and define a discrete interval as follows :

$$[0,1]_h=[0,1]\cap R_h$$

For a function *f* and three disjoint points x_1, x_2, x_3 in its dominant, the first and second decided difference are defined by

$$[x_1,x_2]f=\frac{f(x_1)-f(x_2)}{x_1-x_2}$$

and

$$[x_1,x_2,x_3]f=\frac{[x_2,x_3]f-[x_1,x_2]f}{(x_3-x_1)}$$

respectively.

For convenience, the write f^2 for $D_h^2 f$ and $f_i^{\{2\}}\ for\ D_h^{\{2\}}\ f(x_i)$ and w(f,p) for the modulus of continuity of f. The discrete norm of a function f over the interval $[0,1]_h$ is defined by

$$\|\, f\,\|' = \max_{x\in[0,1]} |\, f(x)\,|$$

without assuming any smoothness condition on the data f, we shall obtain in the following the bounds for the error function over the discrete interval $[0,1]_h$.

Theorem 3.1 : Suppose s(x, h) is the discrete quartic spline interpolant of theorem 2.1, then

$$\|\, e_i^{\{2\}}\,\| \le C_1(h)k(p,h)w(f,p) \tag{3.1}$$

$$|\, e_i^{\{1\}}\,\| \le C_2(h)k_1(p,h)w(f,p) \tag{3.2}$$

and $\| e(x) \| \le p^2 k*(P,h)\, w(f,p)$ (3.3)

where $k(p,h), k_1(p,h) \& k*(p,h)$ are positive constant of p and h

Proof of theorem 3.1. To obtain the error estimate (3.1) first we replace

$m_1(h)$ by $e^{\{2\}}(x_i) = D_n^{\{2\}} s(x_i,h) - f_i^{\{2\}} = L_i$ Say (3.4)

To estimate row max norm of the matrix L_i in (3.4), we shall need the following lemma due to Lyche [5].

Lemma 3.1. Let $\{a_i\}_{i=1}^m$ and $\{b_j\}_{j-1}^n$ be a given sequence of non-negative real numbers such that $\sum a_i = \sum b_j$, then for any real valued function f defined on a discrete interval $[0,1]_h$, we have

$$\left|\sum_{i=1}^m a_i \left[x_{i0}, x_{i1} - x_{ik}\right]_f - \sum_{j=1}^n b_j [y_{j0}, y_{j1} - y_{jk}]_f \right| \le w(f^{(k)}, |1-p|) \sum a_i / k!$$

where $x_{ik}, y_{jk} \in [0,1]_h$ for relevant values of i, j and k. It may be observed that the right hand side (3.4) is written as

$$|(L_i)| = \left| \sum a_i [x_{i0}, x_{i1}]_f - \sum b_j [y_{j0}, y_{j1}]_f \right| \quad (3.5)$$

$$a_5 = b_5 = a_1 = \frac{1}{p^2}\left[\left(12h^2 + 9P^2\right) - 48h^2\right)\right] = b_1$$

$$a_6 = b_6 = a_2 = -\left[(2h^2+1)P^2 + 1(-7+4h^2)h^2\right] = b_2$$

$$a_7 = b_7 = a_3 = \left[(2h^2+4)P^2 + (17+4h^2)h^2\right] = a_3$$

$$a_4 = \frac{1}{p^2}\left[(24h^2+30)P^2\right] = b_4$$

and $x_{10} = \alpha_{i-1} = y_{10} = x_{40}$

$x_{11} = x_i = x_{41}, y_{11} = x_{i-1}$

$x_{20} = x_{i-1} - h$
$y_{10} = x_i$

$x_{21} = x_i$
$y_{11} = x_{i-1} + h$

$x_{30} = x_i$
$y_{30} = x_i - h$

$x_{31} = x_1 + h$
$x_{31} = x_i$

$y_{40} = \alpha_i$
$y_{41} = x_i$

$x_{50} = \alpha_i, x_{51} = x_{i+1}, y_{50} = \alpha_i, y_{51} = x_i$

$x_{60} = x_i - h, x_{61} = x_i, y_{60} = x_i, y_{61} = x_i + h$

$x_{70} = x_{i+1}, x_{71} = x_{i+1} + h, y_{70} = x_{i+1} - h$

$= y_{71}$

Clearly in (3.5) $\{a_i\}$ and $\{b_j\}$ are sequences of non-negative real numbers such that

$$\sum_{i-1}^7 a_i = \sum_{j=1}^7 b_j = N(P,h)$$
(Say)

Thus applying Lemma 3.1 in (3.5) for i=7=j and k=1, we get

$$\|(L_i)\| \le N(P,h)\, w(f^{\{1\}}, |P|) \quad (3.6)$$

Now using the equation (2.2) and (3.6) in (3.3) we get

$$\| e^{\{2\}}(x_1) \| \le C_1(h) K(P,h)\, w(f^{\{1\}}, P) \quad (3.7)$$

where K(P,h) is some positive function of P and h.

We next to proceed to obtain a upper bound for $e(x)$, replacing $m_i(h)$ by $e_i^{\{2\}}$ in equation (2.2), we obtain

$$e(x,h)=P^2[Q_4(t)e_i^{\{2\}}(x_{i-1})+Q_5(t)e_{i+1}^{\{2\}}(x_{i+1})+M_i(f) \quad (3.8)$$

Now we write M, (f) in term of divided difference as following :

$$M_i(f)=[u_i[x_{i0},x_{i1}]]-[v_j[y_{j0},y_{j1}]]f \quad (3.9)$$

Where $u_1=\frac{P}{6A}\left[(12h^2+9)t-24t^2h^2-48t^3+24t^4\right]=v_1$

$$u_2=\frac{P^2}{6A}\left[(2h^2+1)t-6h^2t^2+(4h^2-7)t^3+6t^4\right]=v_2$$

$$u_3=\frac{P^2}{6A}\left[-(2h^2+4)t+3(5+2h^2)t^2-(17+4h^2)t^3+6t^4\right]=v_3$$

$$u_4=\frac{P}{6A}\left(24h^2+30\right)t=v_4$$

and $x_{10}=x_i, x_{11}=\alpha_i, y_{10}=\alpha_i, y_{11}=x_{i+1}$

$x_{20}=x_i, x_{21}=n_i+h, y_{20}=x_i-h$

$=y_{21}$

$x_{30}=x_{i+1}, x_{31}=x_{i+1}+h, y_{30}=x_{i+1}-h$

$=y_{31}$

$x_{40}=\alpha_i, x_{41}=x_i, y_{40}=x_i, y_{41}=x$

Clearly observing that

$$\sum_{i=1}^{4}u_i=\sum_{j=1}^{4}v_j=\frac{p}{6A}[(36h^2+39)t-24t^2h^2$$

$$-48t^3+24t^4]+\frac{p^2}{6A}[-3t+15t^2-24t^3+12t^4]$$

$=N^*(p,h)$ Say

We again applying Lemma 3.1 in (3.9) for i=j=4 and k=1 to see that

$$|M_i(f)|\leq N^*(P,h)\,w(f^{\{1\}},P) \quad (3.10)$$

Thus using (3.7) and (3.11) in (3.8) we get the following :

$$\|e(x)\|\leq P\,K^*(P,h)\,w(f^{\{1\}},P) \quad (3.11)$$

Where K* (P,h) is a positive constant of P and h, this is the inequality (3.3) of Theorem (3.1) .

We now proceed to obtain an upper bound of $e_i^{\{1\}}$, from equation (2.4) we get

$$s^{\{1\}}{}_i(x,h)=f_iQ_i^{\{1\}}(t)+f_{i+1}Q_2^{\{1\}}(t)+f_{\alpha_i}Q_3^{\{1\}}(t)$$
$$+P^2s_i^{\{2\}}(x,h)Q_4^{\{1\}}(t)+P^2s_{i+1}^{\{2\}}(x,h)Q_5^{\{1\}}(t) \quad (3.12)$$

Thus

$$6Ae_i^{\{1\}}(x,h)=P^2[e_i^{\{2\}}Q_4^{\{1\}}(t)+e_{i+1}^{\{2\}}Q_i^{\{1\}}(t)]+U_i(f) \quad (3.13)$$

Where $U_i(f)=f_iQ_1^{\{1\}}(t)+f_{i+1}Q_2^{\{1\}}(t)+f_{\alpha_i}Q_3^{\{1\}}(t)$

$$+P^2\left[f_i^{\{2\}}Q_4^{\{1\}}(t)+f_{i+1}^{\{2\}}Q_5^{\{1\}}(t)\right]-6A\,f_i^{\{1\}}(x,h)$$

By using Lemma 3.1 and first and second divided difference in $U_i(f)$ as follows:-

$$|U_i(f)|\leq W\left(f^{\{1\}},P\right)\sum_{i=1}^{4}a_i=\sum_{j=1}^{4}b_j \quad (3.14)$$

$$=p\ \left[g(36,39)-48h^2Z-48g(3,1)+96Z(Z^2+h^2)\right]$$

$$+\, p^2\left[-3+30Z-24(3Z^2+h^2)+48Z(Z^2+h^2)\right]$$

Where

$$a_1 = p\ \left[g(12,9)-48^2 Z-48(3Z^2+h^2)+96Z(Z^2+h^2)\right]=b_1$$

$$a_2 = p\ g(24,30)=b_2$$

$$a_3 = p^2\left[-g(2,4)+g(12,30)Z-g(4,17)(3Z^2+h^2)+24Z(Z^2+h^2)\right]=b_3$$

$$a_4 = p^2\left[g(2,1)-12h^2Z+g(4,-7)(3Z^2+h^2+24Z(Z^2+h^2)\right]=b_4$$

and $x_{10}=x_i=x_{20}=x_{30}=y_{31}, x_{11}=\alpha_i=x_{21}=y_{11}$

$y_{10}=x_{i+1}=x_{40}=y_{41},\ y_{20}=x,$

$y_{21}=x+h, x_{31}=x_i+h, y_{30}=x_i-h$

$x_{41}=x_{i+1}+h, y_{40}=x_{i+1}-h$

From equation (3.7) put value of $e_i^{\{2\}}$ in (3.13) we get upper bound of $e_i^{\{1\}}$. This is inequality (3.2) of theorem 3.1.

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