ISSN-2230-9659

Multiple Zeta Values

Vanishree A
Research Scholar, Singhania University
Rajasthan, INDIA

REVIEW ARTICLE

To understand the arithmetic nature of special values of the Riemann zeta function, it has become increasingly clear that multiple zeta values (MZV's for short) must be studied. These are defined as follows :

$$\zeta\left(a_1 \ldots, a_k\right) = \sum_{n_1 > n_2 > \ldots n_k > 0} \frac{1}{n_2^{a_1} n_2^{a_2} \ldots n_k^{a_k}},$$

where $a_1, a_2, \ldots, a_k$ are positive integers with the proviso that $a_1 \neq 1.$ The last condition is imposed to ensure convergence of the series.

There are several advantages to introducing these multiple zeta functions. First, they have an algebraic structure which we describe. It is easy to see that

$$\zeta\left(s_1\right)\zeta\left(s_2\right) = \zeta\left(s_1, s_2\right) + \zeta\left(s_2, s_1\right) + \zeta\left(s_1 + s_2\right)$$

ISSN-2230-9659

Indeed, the left hand side can be decomposed as

$$\sum_{n_1,n_2}\frac{1}{n_1^{s_1}n_2^{s_2}}=\sum_{n_1>n_2}\frac{1}{n_1^{s_1}n_2^{s_2}}+\sum_{n_2>n_1}\frac{1}{n_1^{s_1}n_2^{s_2}}+\sum_{n_1=n_2}\frac{1}{n_1^{s_1}n_2^{s_2}}$$

from which the identity becomes evident. In a similar way, one can show that $\zeta(s_1)\zeta(s_2,\ldots,s_t)$ is again an integral linear combinationof multiple zeta values (MZV's). More generally, the product of any two MZV's as an integral linear combination of MZV's. These identities lead to new relations, like $\zeta(2,1)=\zeta(3)$, an identity which appears in $Ape'ry's$ proof of the irrationality of $\zeta(3)$.

If we let A_r be the Q-vector space spanned by

$\zeta(s_1,s_2,\ldots,s_k)$

with $s_1+s_2+\ldots+s_k = r$, then the product formula for MZV's shows that

$A_r A_s \subseteq A_{r+s}$.

In this way, we obtain a graded algebra of MZV's. Let d_r be the dimension of A_r as a vector space over Q. For convenience, we set $d_0 = 1$ and $d_1 = 0$. Clearly, $d_2 = 1$ since A_2 is spanned by $\pi^2/6$. Zagier has made the following conjecture: $d_r = d_{r-2} + d_{r-3}$, for $r \geq 3$. In other words, d_r satisfies a Fibonacci-type recurrence relation. Consequently, d_r is expected to have exponential growth. Given this prediction, it is rather remarkable that not a single value of *r* is knows for which $r_r \geq 2!$.

In view of the identity, $\zeta(2,1)=\zeta(3)$, we see that $d_3 = 1$. What about d_4? A_4 is spanned by $\zeta(4),\zeta(3,1),\zeta(2,2),\zeta(2,1,1)$. What are these numbvers? Zagier's conjecture predicts that $d_4 = d_2 + d_1 = 1 + 0 = 1$. Is this true?

Let's adapt Euler's technique to evaluate $\zeta(2,2)$. As noted in the introduction,

ISSN-2230-9659

$(1 - x/r_1)\ (1 - x/r_2)\ \dots\ (1 - x/r_n)$

has roots equal to $r_1, r_2, \dots, r_n$. When we expand the polynomial, the coeffcient of x is

$-(1/r_1 + 1/r_2 + \dots + 1/r_n)$.

The coefficient of x^2 is

$$\sum_{i<j} 1/r_i r_j .$$

With this observation, we see from the product expansion

$$f(x) = \frac{\sin \pi x}{\pi x} = (1 - x^2)(1 - x^2/4)(1 - x^2/9)..$$

that the coefficient of x^4 is preciesely $\zeta(2,2)$. An easy computation shows that

$$\zeta(2,2) = \pi^4/5!.$$

It is now clear that this method can be used to evaluate $\zeta(2,2,\dots,2) = \zeta(\{2\}^m)$ (say). By comparing the coefficient of x^{2m} in our expansion of $f(x)$, we obtain that

$$\zeta(\{2\}^m) = \frac{\pi^{2m}}{(2m+1)!}.$$

We could have also evaluated $\zeta(2,2)$ using the identity

$$\zeta(2)^2 = 2\zeta(2,2) + \zeta(4),$$

but we had opted to the method above to indicate its generalization which allows us to also evaluate $\zeta(2,2,\dots,2)$. What about $\zeta(3,1)$? This is a bit more difficult and will not come out of out earlier work. In 1998, Borwein, Bradley, Brodhurst and Lisonek showed that $\zeta(3,1) = 2\pi^4/6!$. What about $\zeta(2,1,1)$? With some work, one can show that this is equal to $\zeta(4)$. Thus, we conclude that d_4=1 as predicted by Zagier.

What about d_5? With more work, we can show that

$$\zeta(2,1,1,1)=\zeta(5); \qquad \zeta(3,1,1)=\zeta(4,1)=2\zeta(5)-\zeta(2)\zeta(3)$$

$$\zeta(2,1,1)=\zeta(2,3)=9\zeta(5)/2-2\zeta(2)\zeta(3)$$

$$\zeta(2,2,1)=\zeta(3,2)=3\zeta(2)\zeta(3)-11\zeta(5)/2.$$

This proves that $d_5 \le 2$. Zagier conjectures that $d_5 = 2$. In other words, $d_5 = 2$ if and only if $\zeta(2)\zeta(3)/\zeta(5)$ is irrational.

Can we prove Zagier's conjecture? To this date, not a single example is known for which $d_n \ge 2$. If we write

$$(1-x^2-x^3)^{-1}=\sum_{n=1}^{\infty} D_n x^n,$$

then it is easy to see that Zagier's conjecture is equivalent to the assertion that $d_n = D_n$ for all $n \ge 1$. Deligne and Goncharov and (independently) Terasoma showed that $d_n \le D_n$.

REFERENCES

1. A. Baker, Transcendental Number Theory, Cambridge University Press, 1975.
2. A. Baker, B. birch and E. Wirsing, On a problem of Chowla, Journal of Number Theory, 5 (1973), 224-236.
3. W. Dunham, Journey Through Genius, 1990, John wiley and Sons.
4. S. Gun, M. Ram Murty and P. Rath, On a conjecture of Chowla and Milnor, to appear.

ISSN-2230-9659

5. S. Lang, Algebra, Third edition, Springer-Verlag, 1993.
6. T. rivoal and W. Zudillin, Diophantine properties of numbers related to Catalan's constant, Math. Ann. 326 (2003), no. 4, 705-721.
7. J. Rotman, A First course in Abstract Algebra, Prentice-Hall, 1996.
8. T. Terasoma, Mixed Tate motives and multiple zeta values, Inventiones Math., 149, (2002), no. 2, 339-369.
9. L. Washington, Introduction to Cyclotomic Fields, Springer-Verlag, 1982.