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REVIEW ARTICLE

CONSTANT COEFFICIENTS LINEAR HIGHER-ORDER DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Constant Coefficients Linear Higher-Order Differential-Algebraic Equations

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INTRODUCTION

In this paper, we consider systems of linear /th-order $(l \geq 2)$ differential-algebraic equations with constant coefficients of the form

$$A_l x^{(l)}(t) + A_{l-1} x^{(l-1)}(t) + \cdots + A_0 x(t) = f(t),\ t \in [t_0, t_1], \quad (2.1)$$

where

$A_i \in \mathbb{C}^{m \times n},\ i = 0, 1, \ldots, l,\ A_l \neq 0,\ f(t) \in \mathcal{C}^{\mu}([t_0, t_1], \mathbb{C}^m),$

possibly together with initial conditions

$$x(t_0) = x_0,\ \ldots,\ x^{(l-2)}(t_0) = x_0^{[l-2]},\ x^{(l-1)}(t_0) = x_0^{[l-1]},\quad x_0, \ldots, x_0^{[l-2]}, x_0^{[l-1]} \in \mathbb{C}^n. \quad (2.2)$$

Here, the nonnegative integer μ is the *strangeness-index* of the system (2.1), i.e., to get continuous solutions of the (2.1), the right-hand side $f(t)$ has to be μ-times continuously differentiable (later, in Section 2.2 we shall give an explicit definition of the strangeness- index).

First, let us clarify the concepts of *solution* of the system (2.1), *solution* of the initial value problem (2.1)-(2.2), and *consistency* of the initial conditions (2.2).

Definition 2.1 A vector-valued function $x(t) := [x_1(t), \ldots, x_n(t)]^T \in \mathcal{C}([t_0, t_1], \mathbb{C}^n)$ *is called* solution of *(2.1) if* $\sum_{k=1}^{n} A_i(j,k) x_k^{(i)}(t),\ i = 0, \ldots, l,\ j = 1, \ldots, m,$ *exist and for j = 1,..., m the following equations are satisfied:*

$$\sum_{k=1}^{n} A_l(j,k) x_k^{(l)}(t) + \sum_{k=1}^{n} A_{l-1}(j,k) x_k^{(l-1)}(t) + \cdots + \sum_{k=1}^{n} A_0(j,k) x_k(t) = f_j(t),$$

where A_j (j, k) denotes the element of the matrix A_j lying on the jth row and the kth column of Aj and f (t) := [fi (t),..., $f_m(t)]^T$.

A vector-valued function $x(t) \in \mathcal{C}([t_0, t_1], \mathbb{C}^n)$ *is called* solution of the initial value problem *(2.1)-(2.2) if it is a solution of (2.1) and, furthermore, satisfies (2.2). Initial conditions (2.2) are called* consistent *with the system (2.1) if the associated initial value problem (2.1)-(2.2) has at least one solution.*

In the last section we saw that DAEs differ in many ways from ordinary differential equations. For instance the circuit in figure 1.3 lead to a DAE where a differentiation process is involved when solving the equations. This differentiation needs to be carried out numerically, which is an unstable operation. Thus there are some problems to be expected when solving these systems. In this section we try to measure the difficulties arising in the theoretical and numerical treatment of a given DAE.

Modelling with differential-algebraic equations plays a vital role, among others, for constrained mechanical systems, electrical circuits and chemical reaction kinetics.

In this section we will give examples of how DAEs are obtained in these fields. We will point out important characteristics of differential-algebraic equations that distinguish them from ordinary differential equations.

More information about differential-algebraic equations can be found in [2, 15] but also in Consider the mathematical pendulum in figure 1.1. By construction the rows of Aa are linearly dependent. However, after deleting one row the remaining rows describe a set of linearly independent equations, The node corresponding to the deleted row will be denoted as the ground node.

As seen in the previous sections a DAE can be assigned an index in several ways. In the case of linear equations with constant coefficients all index notions coincide with the Kronecker index. Apart from that, each index definition stresses different aspects of the DAE under consideration. While the differentiation index aims at finding possible reformulations in terms of ordinary differential equations, the tractability index is used to study DAEs without the use of derivative arrays. In this section we made use of the sequence (3.2) established in the context of the tractability index in order to perform a refined analysis of linear DAEs with properly stated leading terms. We were able to find explicit expressions of (3.12) for these equations with index 1 and 2. Let m be the pendulum's mass which is attached to a rod of length l [15]. In order to describe the pendulum in Cartesian coordinates we write down the potential energy U(x; y) = mgh = mgl ¡ mgy where ¡ x(t); y(t) ¢ is the position of the moving mass at time t. The earth's acceleration of gravity is given by g, the pendulum's height is h. If we denote derivatives of x and y by $x^{\cdot}$ and $y^{\cdot}$ respectively, the kinetic energy Some additional simple examples:

Consider the (linear implicit) DAE system:

Ey' = A y + g(t) with consistent initial conditions and apply implicit Euler:

$E(y_{n+1} - y_n)/h = A\, y_{n+1} + g(t_{n+1})$

and rearrangement gives:

$y_{n+1} = (E - A\, h)^{-1} [E\, y_n + h\, g(t_{n+1})]$

Now the true solution, y(tn), satisfies:

$E[(y(t_{n+1}) - y(t_n))/h + h\, y''(x)/2] = A\, y(t_{n+1}) + g(t_{n+1})$

and defining en = y(tn) - yn, we have:

$e_{n+1} = (E - A\, h)^{-1} [E\, e_n - h_2\, y''(x)/2]$

$e_0 = 0$, known initial conditions

where the columns of Aa correspond to the voltage, resistive and capacitive branches respectively. The rows represent the network's nodes, so that ¡1 and 1 indicate the nodes that are connected by each branch under consideration. Thus Aa assigns a polarity to each branch.

This detailed analysis lead us to results about existence and uniqueness of solutions for DAEs with low index. We were able to figure out precisely what initial conditions are to be posed, namely $D(t_0)x(t_0) = D(t_0)x_0$ and $D(t_0)P_1(t_0)x(t_0) = D(t_0)P_1(t_0)x_0$ in the index 1 and index 2 case respectively.

These initial conditions guarantee that solutions u of the inherent regular ODE (3.5) and (3.10) lie in the corresponding invariant subspace. Let us stress that only those solutions of the regular inherent ODE that lie in the invariant subspace are relevant for the DAE. Even if this subspace varies with t we know the dynamical degree of freedom to be rankG0 and rankG0+rankG1¡m for index 1 and 2 respectively

Based upon these concepts, we are naturally interested in the following questions:

1 Does the behaviour of the system (2.1) differ from that of a system of first-order Differential-Algebraic EquationsSs into which (2.1) may be transformed in the same way as in the classical theory of ODEs?

2 Does the system (2.1) always have solutions? If it has, how many solutions do exist? Under which conditions does it have unique solutions?

3 If the system (2.1) has solutions, how smooth is the right-hand side f (t) required to be?

4 Which conditions are required of consistent initial conditions?

5 Under which conditions does the initial value problem (2.1)-(2.2) have unique solutions?

In the following sections we shall answer the above questions one by one. In Section 2.2 we present an example to show the difference that may occur, in terms of *strangeness-index,* between the higher-order system (2.1) and a system of first-order Differential-Algebraic EquationsSs into which the original system is converted. In Section 2.3 we shall give a condensed form, under *strong equivalence transformations,* for matrix triples that are associated with systems of second-order Differential-Algebraic EquationsSs. Then, in Section 2.4, based on the condensed form, we partially read off the the properties of the corresponding system of second-order Differential-Algebraic EquationsSs, and by *differentiation-and-elimination steps* reduce the system to a simpler but equivalent system. After an inductive procedure of this kind of reduction, we shall present a final equivalent *strangeness-free* system by which we can answer the questions posed in the above. Finally, in Section 2.5, the main results of second-order systems obtained in Section 2.4 are extended to general higher-order systems, and moreover, the connection between the solution behaviour of a system of Differential-Algebraic EquationsSs and *regularity* or *singularity* of the matrix polynomial associated with the system is presented. It is well known that one of the key aspects in which a system of Differential-Algebraic EquationsSs differs from a system of ODEs is that, to get the solutions of Differential-Algebraic EquationsSs, only continuity of the right-hand side f (t) may not be sufficient and therefore higher derivatives of f (t) may be required. Later, in Section 2.4, we will clearly see the reason for this difference.

Definition 2.2 Provided that the system (2.1) has solutions, the minimum number ^ of times that all or part of the right-hand side f (t) *in the system (2.1) must be differentiated in order to determine any solution* x(t) *as a continuous function of* t *is the* strangeness-index *of the system (2.1) of Differential-Algebraic EquationsSs.*

Obviously, according to Definition 2.2, both a system of ODEs and a system of purely algebraic equations have a zero strangeness-index.

In the following, we present an example of an initial value problem for linear second- order Differential-Algebraic EquationsSs to demonstrate the possible difference of strangeness index of the original system from that of the converted first-order system of Differential-Algebraic EquationsSs.

Example 2.3 We investigate the initial value problem for the linear second-order constant coefficient Differential-Algebraic EquationsSs

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \ddot{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) = f(t), \quad t \in [t_0, t_1] \tag{2.3}$$

where $x(t) = [x_1(t), x_2(t)]^T$, and $f(t) = [f_1(t), f_2(t)]^T$ is su_ciently smooth, together with the initial conditions

$$x(t_0) = x_0, \quad \dot{x}(t_0) = x_0^{[1]}, \tag{2.4}$$

where $x_0 = [x_{01}, x_{02}]^T \in \mathbb{C}^2$, $x_0^{[1]} = [x_{01}^{[1]}, x_{02}^{[1]}]^T \in \mathbb{C}^2$.
A short computation shows that system (2.3) has the unique solution

$$\begin{cases} x_1(t) = f_2(t), \\ x_2(t) = f_1(t) - \dot{f}_2(t) - \ddot{f}_2(t). \end{cases} \tag{2.5}$$

Moreover, (2.5) is the unique solution of the initial value problem (2.3)-(2.4) if the initial conditions (2.4) are consistent, namely,

$$\begin{cases} x_{01} &= f_2(t_0), \\ x_{02} &= f_1(t_0) - \dot{f}_2(t_0) - \ddot{f}_2(t_0), \\ x_{01}^{[1]} &= \dot{f}_2(t_0), \\ x_{02}^{[1]} &= \dot{f}_1(t_0) - \ddot{f}_2(t_0) - \left.\frac{d^3 f_2(t)}{dt^3}\right|_{t_0+}. \end{cases} \tag{2.6}$$

If we let

$$v(t) = [v_1(t), v_2(t)]^T = [\dot{x}_1(t), \dot{x}_2(t)]^T, \; y(t) = [v_1(t), v_2(t), x_1(t), x_2(t)]^T,$$

then we have the following initial-value problem for the linear first-order Differential-Algebraic EquationsSs

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dot{y}(t) + \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} y(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ 0 \\ 0 \end{bmatrix}, \tag{2.7}$$

together with the initial condition

$$y(t_0) = [x_{01}^{[1]}, x_{02}^{[1]}, x_{01}, x_{02}]^T.$$

It is immediate that the system (2.7) of first-order Differential-Algebraic Equations has the unique solution

$$\begin{cases} x_1(t) = f_2(t), \\ x_2(t) = f_1(t) - \dot{f}_2(t) - \ddot{f}_2(t), \\ v_1(t) = \dot{f}_2(t), \\ v_2(t) = \dot{f}_1(t) - \ddot{f}_2(t) - f_2^{(3)}(t). \end{cases} \tag{2.9}$$

In this form, (2.9) is the unique solution of the initial value problem (2.7)-(2.8) if the initial condition (2.8) is consistent, i.e.,

$$\begin{cases} x_{01} &= f_2(t_0), \\ x_{02} &= f_1(t_0) - \dot{f}_2(t_0) - \ddot{f}_2(t_0), \\ x_{01}^{[1]} &= \dot{f}_2(t_0), \\ x_{01}^{[1]} &= \dot{f}_1(t_0) - \ddot{f}_2(t_0) - f_2^{(3)}(t_0). \end{cases} \tag{2.10}$$

Remark 2.4 Example 2.3 shows that the second-order system (2.3) has a unique continuous solution (2.5) if and only if the right-hand side satisfies

$$f(t) \in C^2([t_0, t_1], \mathbb{C}^2),$$

whereas the converted first-order system (2.7) has a unique continuous solution if and only if $f(t) \in C^3([t_0, t_1], \mathbb{C}^2)$; or in other words, the *strangeness-index* of the converted first-order system (2.7) is larger by one than that of the original second-order system (2.3). For a general system of /-th-order Differential-Algebraic EquationsSs, it is not difficult to find similar examples.

Differential-Algebraic EquationsSs into an associated system of first-order Differential-Algebraic EquationsSs is not always equivalent in the sense that higher degree of the smoothness of the right-hand side f (t) may be involved in the solutions of the latter.

It should be noted that Example 2.3 also shows that, to obtain continuous solutions of a system of Differential-Algebraic EquationsSs, some parts of the right-hand side f (t) may be required to be more differentiable than other parts which may be only required to be continuous; for a detailed investigation, we refer to, for example, [2, 37, 38]. Nonetheless, in order to simplify algebraic forms of a system of Differential-Algebraic EquationsSs, we usually apply algebraic equivalence transformation to its matrix coefficients. For this reason and to avoid becoming too technical, we always consider the differentiability of the right-hand side vector-valued function f (t) as a whole, and do not distinguish the degrees of smoothness required of its components from each other.

REFERENCES

[1] U. M. Ascher, L. R. Petzold. Projected collocation for higher-order higher-index di_erential-algebraic equations. J. Comp. Appl. Math. 43 (1992) 243{259.

[2] K. Balla, R. M□arz. A uni_ed approach to linear di_erential algebraic algebraic equations and their adjoints. Z. Anal. Anwendungen, 21(2002)3, 783-802.

[3] K. E. Brenan, S. L. Campbell, and L. R. Petzold. Numerical Solutions of Initial- Value Problems in Di_erential-Algebraic Equations. Classics in Applied Mathematics, Vol. 14, SIAM, 1996.

[4] R. Byers, C. He, V. Mehrmann. Where is the nearest non-regular pencil? Lin. Alg. Appl., 285: 81-105, 1998.

[5] E. A. Coddington, R. Carlson. Linear Ordinary Di_erential Equations. SIAM. Philadelphia. 1997.

[6] R. Courant, F. John. Introduction to Calculus and Analysis I. Springer-Verlag, New York, Inc. 1989.

[7] S. L. Campbell. Singular Systems of Di_erential Equations. Pitman, Boston, 1980.

[8] S. L. Campbell. Singular Systems of Di_erential Equations II. Pitman, Boston, 1982.

[9] E. K.-W. Chu. Perturbation of eigenvalues for matrix polynomials via the Bauer{ Fike theorems. SIAM J. Matrix Anal. Appl. 25(2003), pp. 551-573.

[10] E. A. Coddington, N. Levinson. Theory of Ordinary Di_erential Equations. McGraw-Hill Book Company, Inc. 1955.

[11] C. De Boor, H. O. Kreiss. On the condition of the linear systems associated with discretized BVPs of ODEs. SIAM J. Numer. Anal., Vol 23, 1986, pp. 936-939.

[12] J. Demmel. Applied Numerical Linear Algebra. SIAM Press. 1997.

[13] J. -P. Dedieu, F, Tisseur. Perturbation theory for homogeneous polynomial eigenvalue problems. Lin. Alg. Appl., 358: 71-74, 2003.

[14] F. R. Gantmacher. The Theory of Matrices. Vol. I. Chelsea Publishing Company, New York, 1959.

[15] F. R. Gantmacher. The Theory of Matrices. Vol. II. Chelsea Publishing Company, New York, 1959.

[16] E. Griepentrog, M. Hanke and R. M□arz. Toward a better understanding of differential algebraic equations (Introductory survey). Seminar Notes Edited by E. Griepentrog, M. Hanke and R. M□arz, Berliner Seminar on Di_erential-Algebraic Equations, 1992. http://www.mathematik.hu-berlin.de/publ/SB-92-1/s Differential-Algebraic Equations.html

[17] I. Gohberg, P. Lancaster, L. Rodman. Matrix Polynomials. Academic Press. 1982.

[18] E. Griepentrog, R. M□arz. Di_erential-Algebraic Equations and Their Numerical Treatment. Teubner-Texte zur Mathematik, Band 88, 1896.

[19] G. H. Golub, C. F. Van Loan. Matrix Computations, Third Edition. The Johns Hopkins University Press. 1996.

[20] N. J. Higham. Matrix nearness problems and applications. In M. J. C. Gover and S. Barnett, editors, Applications of Matrix Theory, pages 1-27, Oxford University Press, 1989.

[21] D. J. Higham, N. J. Higham. Structured backward error and condition of generalized eigenvalue problems. SIAM J. Matrix Anal. Appl. 20 (2) (1998) 493-512.

[22] N. J. Higham, F. Tisseur. More on pseudospectra for polynomial eigenvalue problems and applications in control theory. Lin. Alg. Appl., 351-352: 435-453, 2002.

[23] N. J. Higham, F. Tisseur. Bounds for Eigenvalues of Matrix Polynomials. Lin. Alg. Appl., 358: 5-22, 2003.

[24] N. J. Higham, F. Tisseur, and P. M. Van Dooren. Detecting a de_nite Hermitian pair and a hyperbolic or elliptic quadratic eigenvalue problem, and associated nearness problems. Lin. Alg. Appl., 351-352: 455-474, 2002.

[25] R. A. Horn, C. R. Johnson. Matrix Analysis. Cambridge University Press, July, 1990.

[26] T.-M. Hwang, W.-W. Lin and V. Mehrmann. Numerical solution of quadratic eigenvalue problems with structure-preserving methods. SIAM J. Sci. Comput. 24:1283-1302, 2003.

[27] P. Kunkel and V. Mehrmann. Smooth factorizations of matrix valued functions and their derivatives. Numer. Math. 60: 115-132, 1991.

[28] P. Kunkel and V. Mehrmann. Canonical forms for linear di_erential-algebraic equations with variable coe_cients. J. Comp. Appl. Math. 56(1994), 225-251.

[29] P. Kunkel and V. Mehrmann. A new look at pencils of matrix valued functions. Lin. Alg. Appl., 212/213: 215-248, 1994.

[30] P. Kunkel and V. Mehrmann. Local and global invariants of linear di_erentialalgebraic equations and their relation. Electron. Trans. Numer. Anal. 4: 138-157, 1996.

[31] P. Kunkel and V. Mehrmann. A new class of discretization methods for the solution of linear di_erential-algebraic equations. SIAM J. Numer. Anal. 33: 1941-1961, 1996.