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REVIEW ARTICLE

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Variable Coefficients Linear Higher-Order Differential-Algebraic Equations

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In this paper, we study linear order differential-algebraic equations with variable coefficients

$$A_l(t)x^{(l)}(t) + A_{l-1}(t)x^{(l-1)}(t) + \cdots + A_0(t)x(t) = f(t),\ t \in [t_0, t_1], \qquad (3.1)$$

where $A_i(t) \in \mathcal{C}([t_0, t_1], \mathbb{C}^{m\times n}),\ i = 0, 1, \ldots, l,$ $A_l(t) \not\equiv 0,\ f(t) \in \mathcal{C}([t_0, t_1], \mathbb{C}^m),$ possibly together with initial conditions

$$x(t_0) = x_0,\ \ldots,\ x^{(l-2)}(t_0) = x_0^{[l-2]},\ x^{(l-1)}(t_0) = x_0^{[l-1]},\quad x_0, \ldots, x_0^{[l-2]}, x_0^{[l-1]} \in \mathbb{C}^n. \qquad (3.2)$$

As in the case of constant coefficients, we shall apply very similar techniques (transforming, differentiating, and inserting) to the system (3.1) with variable coefficients, and obtain parallel results on the system (3.1), and on the initial value problem (3.1)-(3.2).

In the last section we saw that DAEs differ in many ways from ordinary differential equations. For instance the circuit in figure 1.3 lead to a DAE where a differentiation process is involved when solving the equations. This differentiation needs to be carried out numerically, which is an unstable operation. Thus there are some problems to be expected when solving these systems. In this section we try to measure the difficulties arising in the theoretical and numerical treatment of a given DAE.

Modelling with differential-algebraic equations plays a vital role, among others, for constrained mechanical systems, electrical circuits and chemical reaction kinetics.

In this section we will give examples of how DAEs are obtained in these fields. We will point out important characteristics of differential-algebraic equations that distinguish them from ordinary differential equations. More information about differential-algebraic equations can be found in [2, 15] but also in Consider the mathematical pendulum in figure 1.1. By construction the rows of Aa are linearly dependent. However, after deleting one row the remaining rows describe a set of linearly independent equations, The node corresponding to the deleted row will be denoted as the ground node.

As seen in the previous sections a DAE can be assigned an index in several ways. In the case of linear equations with constant coefficients all index notions coincide with the Kronecker index. Apart from that, each index definition stresses different aspects of the DAE under consideration. While the differentiation index aims at finding possible reformulations in terms of ordinary differential equations, the tractability index is used to study DAEs without the use of derivative arrays. In this section we made use of the sequence (3.2) established in the context of the tractability index in order to perform a refined analysis of linear DAEs with properly stated leading terms. We were able to find explicit expressions of (3.12) for these equations with index 1 and 2. Let m be the pendulum's mass which is attached to a rod of length l [15]. In order to describe the pendulum in Cartesian coordinates we write down the potential energy U(x; y) = mgh = mgl ¡ mgy where ¡ x(t); y(t) ¢ is the position of the moving mass at time t. The earth's acceleration of gravity is given by g, the pendulum's height is h. If we denote derivatives of x and y by $x^{\cdot}$ and $y^{\cdot}$ respectively, the kinetic energy Some additional simple examples:

Consider the (linear implicit) DAE system:

Ey' = A y + g(t) with consistent initial conditions and apply implicit Euler:

$E(y_{n+1} - y_n)/h = A\, y_{n+1} + g(t_{n+1})$

and rearrangement gives:

$y_{n+1} = (E - A\, h)^{-1} [E\, y_n + h\, g(t_{n+1})]$

Now the true solution, y(tn), satisfies:

$E[(y(t_{n+1}) - y(t_n))/h + h\, y''(x)/2] = A\, y(t_{n+1}) + g(t_{n+1})$

and defining en = $y(t_n) - y_n$, we have:

$e_{n+1} = (E - A\, h)^{-1} [E\, e_n - h^2\, y''(x)/2]$

$e_0 = 0$, known initial conditions

where the columns of Aa correspond to the voltage, resistive and capacitive branches respectively. The rows represent the network's nodes, so that ¡1 and 1 indicate the nodes that are connected by each branch under consideration. Thus Aa assigns a polarity to each branch.

This detailed analysis lead us to results about existence and uniqueness of solutions for DAEs with low index. We were able to figure out precisely what initial conditions are to be posed, namely $D(t_0)x(t_0) = D(t_0)x_0$ and $D(t_0)P_1(t_0)x(t_0) = D(t_0)P_1(t_0)x_0$ in the index 1 and index 2 case respectively.

These initial conditions guarantee that solutions u of the inherent regular ODE (3.5) and (3.10) lie in the corresponding invariant subspace. Let us stress that only those solutions of the regular inherent ODE that lie in the invariant subspace are relevant for the DAE. Even if this subspace varies with t we know the dynamical degree of freedom to be rankG0 and rankG0+rankG1¡m for index 1 and 2 respectively Analogous to Section 2.3, in Section 3.1 we concentrate on the treatment of linear second-order Differential-Algebraic Equations with variable coefficients. We shall prove that the quantities developed in Section 2.3 are still invariant under *local equivalence transformations,* and present a condensed form under a set of *regular conditions.* Later, in Section 3.2, based on the results of Section 3.1, we describe the solution behavior (solvability, uniqueness of solutions and consistency of initial values) of the higher-order system (3.1) and of the initial value problem (3.1)-(3.2).

It should be pointed out that the work in this chapter is carried out along the lines of the work with respect to linear first-order Differential-Algebraic Equations with variable coefficients in [28, 29, 34]; for a comprehensive exposition, we refer to [34], Chapter 3.

Triples of Matrix-Valued Condensed Form Functions

In this section, we shall mainly treat systems of linear second-order Differential-Algebraic Equations with variable coefficients

$$M(t)\ddot{x}(t)+C(t)\dot{x}(t)+K(t)x(t)=f(t),\quad t\in[t_0,t_1], \tag{3.3}$$

where $M(t), C(t), K(t) \in \mathcal{C}([t_0,t_1],\mathbb{C}^{m\times n}),\ f(t)\in\mathcal{C}([t_0,t_1],\mathbb{C}^{m}),$
possibly together with initial value conditions

$$x(t_0)=x_0,\ \dot{x}(t_0)=x_0^{[1]},\quad x_0,x_0^{[1]}\in\mathbb{C}^n. \tag{3.4}$$

we consider the time varying coordinate transformations given by x(t) = *Q(t)y(t)* and premultiplication by P(t), where $Q(t)\in\mathcal{C}^2([t_0,t_1],\mathbb{C}^{n\times n})$ and $P(t)\in\mathcal{C}([t_0,t_1],\mathbb{C}^{m\times m})$ are pointwise nonsingular matrix-valued functions. These changes of coordinates transform (3.3) to an *equivalent* system of Differential-Algebraic Equations

$$\begin{aligned}
&\tilde{M}(t)\ddot{y}(t)+\tilde{C}(t)\dot{y}(t)+\tilde{K}(t)y(t)\\
:=\ &P(t)M(t)Q(t)\ddot{y}(t)+\left(P(t)C(t)Q(t)+2P(t)M(t)\dot{Q}(t)\right)\dot{y}(t)\\
&+\left(P(t)K(t)Q(t)+P(t)C(t)\dot{Q}(t)+P(t)M(t)\ddot{Q}(t)\right)y(t)=P(t)f(t).
\end{aligned} \tag{3.5}$$

Obviously, the relation

$$x(t)=Q(t)y(t)\ (\text{or } y(t)=Q^{-1}(t)x(t))$$

gives a one-to-one correspondence between the two corresponding solution sets of the system (3.3) and the system (3.5). If we use the notation of triples (M(t), C(t), K(t)) and $(\tilde{M}(t),\tilde{C}(t),\tilde{K}(t))$ to represent the systems (3.3) and (3.5) respectively, then we can write the *equivalent relation* in terms of matrix multiplications:

$$[\tilde{M}(t),\tilde{C}(t),\tilde{K}(t)]=P(t)[M(t),C(t),K(t)]\begin{bmatrix}Q(t) & 2\dot{Q}(t) & \ddot{Q}(t)\\ 0 & Q(t) & \dot{Q}(t)\\ 0 & 0 & Q(t)\end{bmatrix}. \tag{3.6}$$

In the general case of order system (3.1), if we make use of the notation of an $(l+1)$-tuple $(A_l(t),\dots,A_1(t),A_0(t))$ of matrix-valued functions to represent the system (3.1), we have the following definition of *equivalence* of variable coefficient systems via time varying transformations.

Definition

3.1 *Two $(l+1)$-tuples $(A_l(t),\dots,A_1(t),A_0(t))$ and $(B_l(t),\dots,B_1(t),B_0(t))$ of matrix-valued functions with* $A_i(t),B_i(t)\in\mathcal{C}([t_0,t_1],\mathbb{C}^{m\times n}),\ i=0,1,\dots,l,$ *are called* (globally) equivalent *if there arepointwise nonsingular matrix-valued functions* $P(t)\in\mathcal{C}([t_0,t_1],\mathbb{C}^{m\times m})$ *and* $Q(t)\in\mathcal{C}^l([t_0,t_1],\mathbb{C}^{n\times n})$ *such that*

$$[B_l(t),\ldots,B_0(t)]$$

$$= P(t)[A_l(t),\ldots,A_0(t)]\begin{bmatrix} Q(t) & \binom{l}{1}\frac{d}{dt}Q(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q(t) \\ & Q(t) & \binom{l-1}{1}\frac{d}{dt}Q(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q(t) & \binom{1}{1}\frac{d}{dt}Q(t) \\ & & & & Q(t) \end{bmatrix}, \quad (3.7)$$

where $\binom{j}{i} = j!/(j-i)!i!$ *denotes a binomial coefficient,* $i, j \in \mathbb{N},\ i \le j$. *If this is the case and the context is clear, we still write* $(A_l(t),\ldots,A_1(t),A_0(t)) \sim (B_l(t),\ldots,B_1(t),B_0(t))$.

As already suggested by the definition, the following proposition shows that relation (3.7) is an equivalence relation.

Proposition 3.2 *Relation (3.7) introduced in Definition 3.1 is an equivalence relation on the set of (l + 1)-tuples of matrix-valued functions.*

Proof. We shall show relation (3.7) has the three properties required of an equivalence relation.

1. Reflexivity: Let $P(t) = I_m$ and $Q(t) = I_n$. Then, we have $(A_l(t),\ldots,A_1(t),A_0(t)) \sim (A_l(t),\ldots,A_1(t),A_0(t))$.

2. Symmetry: Assume that $(A_l(t),\ldots,A_1(t),A_0(t)) \sim (B_l(t),\ldots,B_1(t),B_0(t))$ with pointwise nonsingular matrix-valued functions P(t) and Q(t) that satisfy (3.7). We shall prove that $(B_l(t),\ldots,B_1(t),B_0(t)) \sim (A_l(t),\ldots,A_1(t),A_0(t))$.

   Note that, from the identity $Q(t)Q^{-1}(t) = I$ it follows that any order derivative of $Q(t)Q^{-1}(t)$ with respect to t is identically zero. Then, by this fact, it is immediate to verify that

$$\begin{bmatrix} Q(t) & \binom{l}{1}\frac{d}{dt}Q(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q(t) \\ & Q(t) & \binom{l-1}{1}\frac{d}{dt}Q(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q(t) & \binom{1}{1}\frac{d}{dt}Q(t) \\ & & & & Q(t) \end{bmatrix}$$

$$\cdot \begin{bmatrix} Q^{-1}(t) & \binom{l}{1}\frac{d}{dt}Q^{-1}(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q^{-1}(t) \\ & Q^{-1}(t) & \binom{l-1}{1}\frac{d}{dt}Q^{-1}(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q^{-1}(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q^{-1}(t) & \binom{1}{1}\frac{d}{dt}Q^{-1}(t) \\ & & & & Q^{-1}(t) \end{bmatrix} = I. \quad (3.8)$$

Hence, by (3.7) and (3.8), we have

$$[A_l(t),\ldots,A_0(t)] = P^{-1}(t)[B_l(t),\ldots,B_0(t)]$$

$$\cdot \begin{bmatrix} Q^{-1}(t) & \binom{l}{1}\frac{d}{dt}Q^{-1}(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q^{-1}(t) \\ & Q^{-1}(t) & \binom{l-1}{1}\frac{d}{dt}Q^{-1}(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q^{-1}(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q^{-1}(t) & \binom{1}{1}\frac{d}{dt}Q^{-1}(t) \\ & & & & Q^{-1}(t) \end{bmatrix},$$

namely,

$$(B_l(t),\ldots,B_1(t),B_0(t)) \sim (A_l(t),\ldots,A_1(t),A_0(t)).$$

3. Transitivity: Assume that $(A_l(t),\ldots,A_0(t)) \sim (B_l(t),\ldots,B_0(t))$ with point wise nonsingular matrix-valued functions $P_1(t)$ and $Q_1(t)$ and that $(B_l(t),\ldots,B_0(t)) \sim (C_l(t),\ldots,C_0(t))$ with point wise nonsingular matrix-valued functions $P_2(t)$ and $Q_2(t)$, which satisfy (3.7), respectively. We shall prove that $(A_l(t),\ldots,A_0(t)) \sim (C_l(t),\ldots,C_0(t))$. By the product rule and Leibniz's rule (cf. [6], p. 203) for differentiation, we can immediately verify that

$$\begin{bmatrix} Q_1(t) & \binom{l}{1}\frac{d}{dt}Q_1(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q_1(t) \\ & Q_1(t) & \binom{l-1}{1}\frac{d}{dt}Q_1(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q_1(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q_1(t) & \binom{1}{1}\frac{d}{dt}Q_1(t) \\ & & & & Q_1(t) \end{bmatrix}$$

$$\cdot \begin{bmatrix} Q_2(t) & \binom{l}{1}\frac{d}{dt}Q_2(t) & \cdots & \cdots & \binom{l}{l}\frac{d^l}{dt^l}Q_2(t) \\ & Q_2(t) & \binom{l-1}{1}\frac{d}{dt}Q_2(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}Q_2(t) \\ & & \ddots & \ddots & \vdots \\ & & & Q_2(t) & \binom{1}{1}\frac{d}{dt}Q_2(t) \\ & & & & Q_2(t) \end{bmatrix} \quad (3.9)$$

$$= \begin{bmatrix} Q_1(t)Q_2(t) & \binom{l}{1}\frac{d}{dt}(Q_1(t)Q_2(t)) & \cdots & \binom{l}{l}\frac{d^l}{dt^l}(Q_1(t)Q_2(t)) \\ & Q_1(t)Q_2(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}(Q_1(t)Q_2(t)) \\ & & \ddots & \vdots \\ & & \ddots & \binom{1}{1}\frac{d}{dt}(Q_1(t)Q_2(t)) \\ & & & Q_1(t)Q_2(t) \end{bmatrix}.$$

Thus, by the assumptions and (3.9), we have

$$[C_l(t),\ldots,C_0(t)] = P_1(t)P_2(t)[A_l(t),\ldots,A_0(t)]$$

$$\cdot \begin{bmatrix} Q_1(t)Q_2(t) & \binom{l}{1}\frac{d}{dt}(Q_1(t)Q_2(t)) & \cdots & \binom{l}{l}\frac{d^l}{dt^l}(Q_1(t)Q_2(t)) \\ & Q_1(t)Q_2(t) & \cdots & \binom{l-1}{l-1}\frac{d^{l-1}}{dt^{l-1}}(Q_1(t)Q_2(t)) \\ & & \ddots & \vdots \\ & & \ddots & \binom{1}{1}\frac{d}{dt}(Q_1(t)Q_2(t)) \\ & & & Q_1(t)Q_2(t) \end{bmatrix},$$

namely,

$$(A_l(t),\ldots,A_1(t),A_0(t)) \sim (C_l(t),\ldots,C_1(t),C_0(t)).$$

In order to introduce a set of *regularity conditions* under which we can get a condensed form via (global) equivalence transformations (3.6) for the triple (M(t), C(t), K(t)) in (3.3), we need the concept of *(local) equivalence relation* between two triples of matrices.

Two triples (M, C, K) and $(\tilde{M},\tilde{C},\tilde{K})$, $M,C,K,\tilde{M},\tilde{C},\tilde{K} \in \mathbb{C}^{m\times n}$, of matrices are called (locally) equivalent if there exist matrices $P \in \mathbb{C}^{m\times m}$ and $Q,A,B \in \mathbb{C}^{n\times n}$, P, Q nonsingular, such that

$$\tilde{M} = PMQ, \quad \tilde{C} = PCQ + 2PMA, \quad \tilde{K} = PKQ + PCA + PMB. \qquad (3.10)$$

In general, we have the following definition of *(local) equivalence relation* between two tuples of matrices.

Definition 3.3 *Two (1+1)-tuples* $(A_l,\ldots,A_1,A_0)$ *and* $(B_l,\ldots,B_1,B_0)$, $A_i, B_i \in \mathbb{C}^{m\times n}$, $i = 0,1,\ldots,l$, $l \in \mathbb{N}_0$, *of matrices are called* (locally) equivalent *if there exist matrices* $P \in \mathbb{C}^{m\times m}$, $Q, R_1,\ldots,R_l \in \mathbb{C}^{n\times n}$, P, Q *nonsingular, such that*

$$[B_l,\ldots,B_0] = P[A_l,\ldots,A_0]\begin{bmatrix} Q & \binom{l}{1}R_1 & \cdots & \cdots & \binom{l}{l}R_l \\ & Q & \binom{l-1}{1}R_1 & \cdots & \binom{l-1}{l-1}R_{l-1} \\ & & \ddots & \ddots & \vdots \\ & & & Q & \binom{1}{1}R_1 \\ & & & & Q \end{bmatrix}, \qquad (3.11)$$

Again, we write $(A_l,\ldots,A_1,A_0) \sim (B_l,\ldots,B_1,B_0)$ *if the context is clear.*

Proposition 3.4 *Relation (3.11) introduced in Definition 3.3 is an equivalence relation on the set of (l + 1)-tuples of matrices.*

Proof. The proof can be immediately carried out along the lines of the proof of Proposition 3.2.

Recalling the condensed form and the invariants for matrix triples obtained under (strong) equivalence transformations in Section 2.3, we can introduce a set of invariants for matrix triples under local equivalence transformations, as the following lemma shows.

Lemma 3.5 *Under the same assumption and the same notation as in Lemma 2.9, the quantities defined in (2.30) are invariant under the local equivalence relation (3.10) and* (M, C, K) *is locally equivalent to the form* (2.28).

Proof. Since the strong equivalence relation (2.17) is the special case of the local equivalence relation (3.11) by setting $R_i = 0, i = 1,\ldots,l$, by Lemma 2.9, it is immediate that (M, C, K) is locally equivalent to the form (2.28). In view of the proof of Lemma 2.9, it remains to show that the quantities defined in (2.30) are invariant under the local equivalence relation (3.10). Here, again, we just take $s^{(MCK)}$ as an example. Indeed, let (M, C, K) and $(\tilde{M},\tilde{C},\tilde{K})$ be locally equivalent, namely, identity (3.10) holds. Let the columns of $\tilde{Z}_1$ form a basis for $\mathcal{N}(\tilde{M}^T)$, and let the columns of $\tilde{Z}_3$ form a basis for $\mathcal{N}(\tilde{M}^T) \cap \mathcal{N}(\tilde{C}^T)$. Then, from (3.10)it follows that the columns of $Z_1 := P^T\tilde{Z}_1$ form a basis for $\mathcal{N}(M^T)$. Since, for any $z \in \tilde{Z}_3$,

$$Q^TM^TP^Tz = 0, \quad Q^TC^TP^Tz + 2A^TM^TP^Tz = 0,$$

if and only if

$$M^TP^Tz = 0, \quad C^TP^Tz = 0,$$

it follows that the columns of $Z_3 := P^T\tilde{Z}_3$ form a basis for $\mathcal{N}(M^T) \cap \mathcal{N}(C^T)$. Thus, the invariance of $s^{(MCK)}$ follows from

$$
\begin{aligned}
\tilde{s}^{(MCK)} &= \dim\left(\mathcal{R}(\tilde{M}^T)\cap\mathcal{R}(\tilde{C}^T\tilde{Z}_1)\cap\mathcal{R}(\tilde{K}^T\tilde{Z}_3)\right)\\
&= \dim\left(\mathcal{R}(Q^TM^TP^T)\cap\mathcal{R}(Q^TC^TP^T\tilde{Z}_1+2A^TM^TP^T\tilde{Z}_1)\right.\\
&\qquad\left.\cap\,\mathcal{R}(Q^TK^TP^T\tilde{Z}_3+A^TC^TP^T\tilde{Z}_3+B^TM^TP^T\tilde{Z}_3)\right)\\
&= \dim\left(\mathcal{R}(Q^TM^TP^T)\cap\mathcal{R}(Q^TC^TP^T\tilde{Z}_1)\cap\mathcal{R}(Q^TK^TP^T\tilde{Z}_3)\right)\\
&= \dim\left(\mathcal{R}(M^TP^T)\cap\mathcal{R}(C^TP^T\tilde{Z}_1)\cap\mathcal{R}(K^TP^T\tilde{Z}_3)\right)\\
&= \dim\left(\mathcal{R}(M^T)\cap\mathcal{R}(C^TZ_1)\cap\mathcal{R}(K^TZ_3)\right)\\
&= s^{(MCK)}.
\end{aligned}
$$

Similarly, the invariance of the other quantities in (2.30) can be proved.

Now, from the matrix triple (M, C, K) passing to the triple (M(t),C(t),K(t)) of matrix-valued functions, we can calculate, based on Lemma 3.5, the characteristic quantities in (2.30) for (M(t),C(t),K(t)) at each fixed value $t \in [t_0, t_1]$. Then, we obtain nonnegative-integer valued functions

$$r, a, s^{(MCK)}, s^{(CK)}, d^{(1)}, s^{(MC)}, s^{(MK)}, d^{(2)}, v, u : [t_0, t_1] \to \mathbb{N}_0.$$

For the triple (M(t),C(t),K(t)) of matrix-valued functions, in order to derive a condensed form which is similar in form to the condensed form (2.28) for the matrix triple (M, C, K), we introduce the following assumption of *regularity conditions* for the triple (M(t),C(t), K(t)) on $[t_0, t_1]$:

$$
\begin{aligned}
&r(t)\equiv r,\ a(t)\equiv a,\ s^{(MCK)}(t)\equiv s^{(MCK)},\ \ s^{(CK)}(t)\equiv s^{(CK)},\\
&d^{(1)}(t)\equiv d^{(1)},\ \ s^{(MC)}(t)\equiv s^{(MC)},\ \ s^{(MK)}(t)\equiv s^{(MK)}.
\end{aligned}
\tag{3.12}
$$

By (2.30) and (3.12), it immediately follows that $d^{(2)}(t), v(t), u(t)$ are also constant on $[t_0, t_1]$.

We can see that the regularity conditions (3.12) imply that the sizes of the blocks in the condensed form (2.28) do not depend on $t \in [t_0, t_1]$. Then, the assumption (3.12) allows for the application of the following property of a matrix-valued function with a constant rank, which may be regarded as a generalization of the property of a matrix shown in Lemma 2.6.

Lemma 3.6 *([34], p. 71) Let* $A(t) \in \mathcal{C}^l([t_0, t_1], \mathbb{C}^{m\times n})$, $l \in \mathbb{N}_0 \cup \{\infty\}$, *and* rank $A(t) \equiv r$, $r \in \mathbb{N}_0$, *for all* $t \in [t_0, t_1]$. *Then there exist pointwise unitary (and therefore non-singular) matrix-valued functions* $U(t) \in \mathcal{C}^l([t_0, t_1], \mathbb{C}^{m\times m})$ *and* $V(t) \in \mathcal{C}^l([t_0, t_1], \mathbb{C}^{n\times n})$, *such that*

$$U^H(t)A(t)V(t) = \begin{bmatrix} \Sigma(t) & 0 \\ 0 & 0 \end{bmatrix}, \tag{3.13}$$

where $\Sigma(t) \in \mathcal{C}^l([t_0, t_1], \mathbb{C}^{r\times r})$ *is nonsingular for all* $t \in [t_0, t_1]$.

Using Lemma 3.6 we can then obtain the following global condensed form for triples of matrix-valued functions via global equivalence transformations (3.6). For convenience of expression, in the following condensed form and its proof, we drop the subscripts of the blocks and omit the argument t unless they are needed for clarification.

Lemma 3.7 *Let* $M(t), C(t), K(t) \in \mathcal{C}([t_0, t_1], \mathbb{C}^{m\times n})$ *be sufficiently smooth, and sup pose that the regularity conditions (3.12) hold for the local characteristic values of* (M(t), C(t), K(t)). *Then,* (M(t),C(t),K(t)) *is globally equivalent to a triple* $(\tilde{M}(t), \tilde{C}(t), \tilde{K}(t))$ *of matrix-valued functions of the following condensed form*

$$
\left(
\begin{bmatrix}
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & I_{s^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & I_{s^{(MK)}} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & I_{d^{(2)}} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & C & C & 0 & 0 & C & C\\
0 & 0 & C & C & 0 & 0 & C & C\\
0 & 0 & C & C & 0 & 0 & C & C\\
0 & 0 & C & C & 0 & 0 & C & C\\
0 & 0 & 0 & 0 & I_{s^{(CK)}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & I_{d^{(1)}} & 0 & 0\\
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & I_{s^{(MC)}} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & K & 0 & K & 0 & K & 0 & K\\
0 & 0 & 0 & 0 & 0 & 0 & I_a & 0\\
0 & 0 & 0 & 0 & I_{s^{(CK)}} & 0 & 0 & 0\\
0 & 0 & I_{s^{(MK)}} & 0 & 0 & 0 & 0 & 0\\
I_{s^{(MCK)}} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\right)
\begin{array}{l}
s^{(MCK)}\\ s^{(MC)}\\ s^{(MK)}\\ d^{(2)}\\ s^{(CK)}\\ d^{(1)}\\ s^{(MCK)}\\ s^{(MC)}\\ a\\ s^{(CK)}\\ s^{(MK)}\\ s^{(MCK)}\\ u
\end{array},
\tag{3.14}
$$

All blocks except the identity matrices in (3.14) are again matrix-valued functions on $[t_0, t_1]$.

Note that $C_{5,4}(t) \equiv 0$ in (3.14) whereas $C_{5,4}$ in (2.28) may be a nonzero matrix, which is the only difference in form between condensed forms (3.14) and (2.28). This difference is due to the equivalence relation (3.5) via time varying transformations. $C_{5,4}(t) \equiv 0$ is obtained by solving an initial value problem for ordinary differential equations; see the details of the proof at the end of page 48.

Proof. The proof of Lemma 3.7 is given in Appendix (on page 48) to this chapter.

The Solution performance of Higher-Order Systems of Differential-Algebraic Equations

Here, the only difference of the case of variable coefficients from the constant case is that, in order to carry out the procedure to the final stage, we must assume that at every stage of the inductive procedure, the regularity conditions (3.12) hold. If this is the case, then it is immediate that we can obtain, finally, a theorem which is parallel to Theorem 2.12. From the final theorem we can directly read off the solution behavior of (3.3) and of (3.3)-(3.4), and obtain a consequence which is parallel to Corollary 2.13. Clearly, there is no difference in form between the final theorem and Theorem 2.12 if in the former case we omit the argument t in the variable coefficients, nor is there between the consequence and Corollary 2.13. Therefore, here, for the sake of space of writing we do not state them again. In addition, it should be pointed out that, at this writing, since we do not know whether two different but globally equivalent triples of matrix-valued functions, after the differentiation-and-elimination steps are applied to them respectively, will lead to new triples with *same* characteristic values $r, a, s^{(MCK)}, s^{(CK)}, d^{(1)}, s^{(MC)}, s^{(MK)}, d^{(2)}, v,$ and u, we can not guarantee that these values obtained in every step of the above inductive procedure are globally characteristic for the triple (M(t),C(t),K(t)). Analogously, in the general case of higher-order systems of Differential-Algebraic Equations with variable coefficients, we can obtain a final theorem which is similar in form to Theorem 2.19, and its consequence similar to Corollary 2.20, which can show the solution behavior of (3.1) and of (3.1)-(3.2). For the same reason, we omit them here.

Appendix: Proof of Lemma 3.7. By the global equivalent relation (3.6) and Lemma 3.6, we obtain the following sequence of globally equivalent triples of matrix- valued functions.

$$(M,C,K)$$

$$\sim\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} C & C \\ C & C \end{bmatrix},\begin{bmatrix} K & K \\ K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix},\begin{bmatrix} C & C \\ C & U_1^H C V_1 \end{bmatrix}+2\begin{bmatrix} I & 0 \\ 0 & U_1^H \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \dot{V}_1 \end{bmatrix},\begin{bmatrix} K & K \\ K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} C & C & C \\ C & I & 0 \\ C & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K \\ K & K & K \\ K & K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} C & 0 & C \\ C & I & 0 \\ C & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K \\ K & K & K \\ K & K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} C & 0 & C \\ 0 & I & 0 \\ C & 0 & 0 \end{bmatrix}+2\begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ -\dot{C} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K \\ K & K & K \\ K & K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} V_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} C & 0 & C \\ 0 & I & 0 \\ U_2^H C V_2 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K \\ K & K & K \\ K & K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} V_{11} & V_{12} & 0 & 0 \\ V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} C & C & 0 & C \\ C & C & 0 & C \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} V_{11} & V_{12} & 0 & 0 \\ V_{21} & V_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & C \\ 0 & C & 0 & C \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & C \\ 0 & C & 0 & C \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & Q_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & C \\ 0 & CQ_1+2\dot{Q}_1 & 0 & C \\ 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \\ K & K & K & K \end{bmatrix}\right)$$

(where pointwise nonsingular matrix-valued function $Q_1(t)$ is chosen as the solution of the initial value problem $\dot{Q}_1(t) = -\frac{1}{2}C_{2,2}(t)Q_1(t)$, $t \in [t_0,t_1]$, $Q_1(t_0)$=I)

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & C & C\\ 0 & 0 & 0 & C & C\\ 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & K & K\\ K & K & K & K & K\\ K & K & K & 0 & K\\ K & K & K & 0 & K\\ K & K & 0 & I & 0\\ K & K & K & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & C & C\\ 0 & 0 & 0 & C & C\\ 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & 0 & K\\ K & K & K & 0 & K\\ K & K & K & 0 & K\\ K & K & K & 0 & K\\ K & K & 0 & I & 0\\ K & K & K & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & C & C\\ 0 & 0 & 0 & C & C\\ 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & 0 & K\\ K & K & K & 0 & K\\ K & K & K & 0 & K\\ K & K & K & 0 & K\\ 0 & 0 & 0 & I & 0\\ K & K & K & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & 0 & C & C\\ 0 & 0 & 0 & 0 & C & C\\ 0 & 0 & V_{11} & V_{12} & 0 & 0\\ 0 & 0 & V_{21} & V_{22} & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & K & 0 & K\\ K & K & K & K & 0 & K\\ K & K & K & K & 0 & K\\ K & K & K & K & 0 & K\\ K & K & K & K & 0 & K\\ 0 & 0 & 0 & 0 & I & 0\\ K & K & I & 0 & 0 & 0\\ K & K & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & 0 & C & C\\ 0 & 0 & 0 & 0 & C & C\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ 0 & 0 & 0 & 0 & I & 0\\ K & K & I & 0 & 0 & 0\\ K & K & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & 0 & C & C\\ 0 & 0 & 0 & 0 & C & C\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ 0 & 0 & 0 & 0 & I & 0\\ K & 0 & I & 0 & 0 & 0\\ K & K & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & 0 & 0 & C & C\\ 0 & 0 & 0 & 0 & C & C\\ 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ K & K & 0 & K & 0 & K\\ 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & I & 0 & 0 & 0\\ K & K & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & V_{11} & V_{12} & 0 & 0 & 0 & 0\\ 0 & V_{21} & V_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & C & 0 & 0 & C & C\\ 0 & 0 & 0 & 0 & 0 & C & C\\ 0 & 0 & 0 & 0 & 0 & C & C\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}+2\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \dot{V}_{11} & \dot{V}_{12} & 0 & 0 & 0 & 0\\ 0 & \dot{V}_{21} & \dot{V}_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\right.$$

$$\begin{bmatrix} K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ 0 & 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ K & I & 0 & 0 & 0 & 0 & 0\\ K & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}+\begin{bmatrix} 0 & C & C & 0 & 0 & C & C\\ 0 & 0 & 0 & 0 & 0 & C & C\\ 0 & 0 & 0 & 0 & 0 & C & C\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \dot{V}_{11} & \dot{V}_{12} & 0 & 0 & 0 & 0\\ 0 & \dot{V}_{21} & \dot{V}_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.+\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \ddot{V}_{11} & \ddot{V}_{12} & 0 & 0 & 0 & 0\\ 0 & \ddot{V}_{21} & \ddot{V}_{22} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & C & 0 & 0 & C & C\\ 0 & C & C & 0 & 0 & C & C\\ 0 & C & C & 0 & 0 & C & C\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ K & K & K & 0 & K & 0 & K\\ 0 & 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ K & I & 0 & 0 & 0 & 0 & 0\\ K & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$\sim\left(\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & I & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & I & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} 0 & C & C & 0 & 0 & C & C\\ 0 & C & C & 0 & 0 & C & C\\ 0 & C & C & 0 & 0 & C & C\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & I & 0 & 0\\ I & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},\begin{bmatrix} K & 0 & K & 0 & K & 0 & K\\ K & 0 & K & 0 & K & 0 & K\\ K & 0 & K & 0 & K & 0 & K\\ K & 0 & K & 0 & K & 0 & K\\ K & 0 & K & 0 & K & 0 & K\\ K & 0 & K & 0 & K & 0 & K\\ 0 & 0 & 0 & 0 & 0 & I & 0\\ 0 & 0 & 0 & I & 0 & 0 & 0\\ K & I & 0 & 0 & 0 & 0 & 0\\ K & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

$$
+2\begin{bmatrix}
\dot{V}_{11} & \dot{V}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot{V}_{21} & \dot{V}_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
+\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot{V}_{11} & \dot{V}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\dot{V}_{21} & \dot{V}_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
+\begin{bmatrix}
\ddot{V}_{11} & \ddot{V}_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
\ddot{V}_{21} & \ddot{V}_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\Bigg)
$$

$$
\sim\left(\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
C & C & C & C & 0 & 0 & C & C \\
C & C & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
K & K & 0 & K & 0 & K & 0 & K \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right)
$$

$$
\sim\left(\begin{bmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & C & C & 0 & 0 & C & C \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix},
\begin{bmatrix}
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & K & 0 & K & 0 & K & 0 & K \\
0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}\right).
$$

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